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Ref: ....

جامعة محمد خيضر بسكرة

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Theme

Quantum Studies of Some Non-Central Potentials

Presented by:

Heddar MEBAREK

Publicly discussed on ....

In front of the Assessment committee consisting of:

Abdellah	ATTAF	Professor	Université de Biskra	President
Mustapha	MOUMNI	Professor	Université de Biskra	supervisor
Mokhtar	FALEK	MCA	Université de Biskra	co supervisor
Achour	BENSLAMA	Professor	Université de Constantine	Examinator
Merad	MAHMOUD	Professor	Université de Oum El Bouaghi	Examinator
Zaiem	SLIMANE	Professor	Université de Batna1	Examinator

Dedication

I dedicate this effort and this work to my parents to my wife and my children Assil,Chem and Ahmed Mejd

Acknowledgment

I would like to express my deep gratitude to my thesis Supervisor, Professor Mustapha Moumni and co Supervisor Mokhtar Falek , for the sense of family that they engendered in their group,for their indefatigable work ethic and boundless curiosity and enthusiasm, fortheir over-lavish encouragement and thoughtfully over-delicate feedback.Thank you to my assessment committee Professors Abdellah Attaf,Achour.Benslama,Merad.Mahmoud and Zaiem Slimane to accept them discussing my thesis.Many thanks go to all the others who've helped and encouragement me over the years my family ,my friends and my colleagues

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0.1 Introduction

It is well established today that quantum mechanics, like other theories, has two aspects, the mathematical and conceptual. ,in the first aspect, it is a consistent and elegant theory and has been immensely successful in explaining and predicting a large number of atomic and subatomic phenomena. But in the second one, it has been a subject of endless discussions without agreed conclusions, without quantum mechanics, it was impossible to understand the enormous phenomena in microscopic physics, which does not appear in our macroscopic world. In this endless way of success for quantum mechanics, mathematics, especially mathematical physics developed to help quantum mechanics.the quantum mechanics give a good results for systems of a particle in a Colombian potential or harmonic potential when the energy and wave function is well defined but these central potentials is not exist in microscopic world but it exist a lot of predict potentials in atomic and subatomic systems some of them have a same proprieties like non-central potentials

Non-central potentials are potentials without spherical symmetry,they don't depend only to the radius (r) but is depend to another parameters like angels, they represent the nature of non-central forces ,this kind of potential take his importance from the real physical systems, such as atoms and molecules, are rarely spherically symmetric such as hydrogen.The study of non-central potential began with the pioneering works of Makarov [1] when he take up the quantum mechanical problem of a particle in the torus shaped potential and after the works of Hartmann [2] in his paper he gave the non-central potentials for which the Schrödinger equation separates in the spherical coordinate and then structured with the work of Hautot [3]. Thus these works have paved the way for more realism in studies,when he was give all non-central potentials which we can solve it analytically in classical mechanics and quantum mechanics when the Newton's and Schrödinger's mechanics are considered ,and he found some exact solutions exist in 2 dimension space and others in 3 dimension space

Hartmann has focused on the ring-shaped potential which called Hartmann potential ,he had many papers in this subject [4] one of them is investigated Spin-Orbit coupling for the motion of a particle in a Ring-Shaped potential.

The non-central potentials especially for which the Schrödinger equation can be solved exactly by separation of variables have been found many applications, particularly in quantum chemistry [5]. They are used to describe the quantum dynamics of ring-shaped molecules like benzene molecule , they have solved the 3 dimension Schrödinger equation by using the Kustaanheimo-Stiefel transformation ,another application of non-central potential is the interactions between deformed nuclei pairs [6] . The potentials without spherical symmetry have some applications within the nanostructure theory [7], and also help us about structuring the metallic glasses [8] . The non-central potentials serve to the theory of the material sciences, for example, describing microscopic elasticity, and obtaining of elastic constants of a cubic crystal [9] .

There are currently a lot of works in the field of non-central, few of them have analytical so-

$f(\theta)$	$V(r)=\frac{H}{r}+\frac{D_r}{r^2}$ Kratzer	$V(r)=kr^2+\frac{D_r}{r^2}$ pseuadoharmonic
$\left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta$	Case1	Case2
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\sin^2\theta+\beta\sin\theta+\gamma\right)\cos^{-2}$	Case3	Case4
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\frac{\theta}{2}+\beta\tan\frac{\theta}{2}+\gamma\right)$	Case5	Case6
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cot^2\frac{\theta}{2}+\beta\cot\frac{\theta}{2}+\gamma\right)$	Case7	Case8
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\theta+\beta\tan\theta+\gamma\right)$	Case9	Case10

Table 1: The solvable non-central potentials in 2 dimensions

lutions and thus they have been studied either with numerical technics or with approximated methods,like the asymptotic iteration method [10] Pekeris approximation [11],factorization method [12],orthogonal polynomial solutions [13],the formalism of supersymmetric quantum mechanics (SUSYQM) [14], Laplace transform approach [15]

The exact analytical solutions of the non-central potentials and their generalizations have been studied in relativistic/non-relativistic regions for many years There are currently a lot of works like [16][17]

To reach our goal this thesis was organized in form with two parts the first part allotted to the study in ordinary space as it contains two chapter ,the first chapter devoted to the study of all solvable non-central potentials $V(r,\theta)=\mu\left[V(r)+\frac{f(\theta)}{r^2}\right]$ in 2D ordinary space,when we have considered the four potentials of Hautot and the dipole potential that appear in the $(Table 1)$

This chapter contain in the first section the nonrelativistic case when we have solved the Schrödinger equation analytically by the separation of variable method to get the energy spectrum and the wave function,also in this section we focused on the dipole,in the two cases with Kratzer and with pseudoharmonic potential where we plotted the energy in terms of the radial and angular momentum then we found a crtical values for this momentum that make the states bounded,moreover,we studied the 2D disc-shaped quantum ring (QR) under the effect of an ionized donor atom quantum where we took the GaAs as an example ,and we plotted the the corrections of the energie due to the dipole the second section is consecrate to studies the relativistic case when we just have considered the spin and pseudo spin limits also in this section we detailed the study of relativistic kratzer +dipole potential and pseudoharmonic dipole too where we found the realivistic energy and we plotted it to show the difference between it and the nonrelativistic energy

In the second chapter we treated all solvable non-central potentials in 3D ordinary space,when we have considered the three potentials of Hautot which appear in the$(Table\ 2)$

In first section we studied it in nonrelativistic case to find the non-relativistic energy and wave function and in the second section we investigated the spin and pseudo spin limits of

$f(\theta)$	$V(r)=\frac{H}{r}+\frac{D_r}{r^2}$ Kratzer	$V(r)=kr^2+\frac{D_r}{r^2}$ pseuadoharmonic
$\left(\frac{\hbar^2}{2\mu^2}\right)\dfrac{(\alpha\cos^2\theta+\beta\cos\theta+\gamma)}{\sin^2\theta}$	hartmann $(\alpha=\beta=0)$ Makarov potential $(\alpha=0)$	hartmann$(\alpha=\beta=0)$ +Harmonic Makarov potential $(\alpha=0)+$ Harmonic
$\left(\frac{\hbar^2}{2\mu^2}\right)\dfrac{(\ \alpha\cos^4\theta+\beta\cos^2\theta+\gamma)}{\sin^2\theta\cos^2\theta}$	ring-shaped potential$(\beta=\gamma=0)$ doublering shaped$(\alpha=0)$	ring-shaped potential $(\beta=\gamma=0)+$ Harmonic doublering shaped $(\alpha=0)+$ Harmonic
$\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cot^2\theta+\beta\cot\theta+\gamma)$	ring-shaped potential$(\alpha=\gamma=0)$	ring-shaped potential $(\alpha=\gamma=0)+$ Harmonic

Table 2: The solvable non-central potentials in 3 dimensions

relativistic case also in this chapter we focused to ring shaped potential ,where we plotted the energy of some levels

In the second part of this thesis we addressed in detail the potentials of the first part in deformed space (de-sitter and anti di-Sitter space) in nonrelativistic case when the deformed energy and deformed wave function are deduced ,this part contain tow chapter the first is devoted to two dimensional deformed space and the second chapter is allotted to the three-dimensional deformed space ,the deformed energy and wave function are deduced ,we focused to the dipole and ring-shaped potential where we plotted the deformed energy in terms of the parameters of deformation ,we found critical values for the parameter of deformation which make the bound states exiset

Part I

The Quantum Studies of Some Non-Central Potentials in Ordinary Space

Chapter 1

Studies of Two Dimensional Non-Central Potentials

1.1 Introduction

Between the complexity of the three dimensional and the simplicity of the one dimensional system the 2D domain attracted the attention of many researchers in several axes in technology ,physics, chemistry and biology. Since the discover of the graphene the 2D matter was be a real and open a big fields of researches .Interest for 2D systems comes from the great popularity of graphene (and co. like Silicene and Manganene) , being one atom-thick carbon nanosheets, became the first 2D nanostructure, which was isolated from parent graphit and also the interest comes from experimental achievements like the motion of the electron around the proton is constrained to be planar (say, by applying a strong magnetic field) then this problem will considered within the context of quantum mechanics as a two-dimensional hydrogen atom. There are many physical applications in which systems are effectively two-dimensional (e.g., adsorbed atoms on surfaces that behave like 2D at low temperatures)with the realization of quantum gases at low dimensions [18][19] and before that from quasi-condensate experiments [20]. furthermore great success has been achieved in nanofabrication techniques in the past decades, especially for the low-semiconductor systems, such as superlattices, quantum well, quantum dots and quantum wires. the immense technological advancement in nano-processing, new beings appear in low dimensional systems like quantum dots (QD) which can be regarded as low-dimensional heterostructures whose carriers are confined in all spatial dimensions [21]. Their manufacturing techniques make it possible to control their properties and thus they are made in such a way that they acquire the same characteristics of atomic systems; this is why they are sometimes called artificial atoms [[21], [22]]. The confinement potential in QD may originate from various physical effects and possesses different symmetries in different nano-structures and the knowledge of realistic profile of confinement potential is necessary for a theoretical description of the electronic properties of QDs and, more importantly, for fabrication of nano-devices [22]Regarding non-central potential in 2D systems there are, the potentials of Hautot which have been solved analytically in nonrelativistic case with the Colombian potential or oscillator potential despite the exact mathematical solution of the Schrödinger equation for these potentials and finding the eigenvalues and the eigenvectors, they remain physically without application in 2D space , and recently, Moumni and Falek were able for the first time to solve the Schrödingar equation analytically for a pure dipole potential where are they found well defined energy and wave function [23],In contrast to the Hauto potentials the pure dipole is present in ultrathin semiconductor layers [24] , in spin-polarized atomic hydrogen absorbed on the surface of superfluid helium [25] , for charged particles in a plane with perpendicular magnetic field [26] and also in gapped graphene with two charged impurities [27] [28] . On the other hand,non-pure dipole potential was recently found in the case of electron pairing that stems from the spin-orbit interaction in two-dimensional quantum well [29].ring-shaped was found in disc-shaped quantum ring (QR) under the effect of an ionized donor atom, the conduction band electron is described by a PHO as a confinement potential and a donor impurity term [[30],[31], [32]].This chapter is divided into two section the first section is devoted to study the analytic solution of $2D$ Schrödinger equation with some non-central potentials which are mentioned in($table$ 1) in ordinary space,in the second section we particularize the same potentials but in the spin and pseudo spin symmetries of relativistic case,

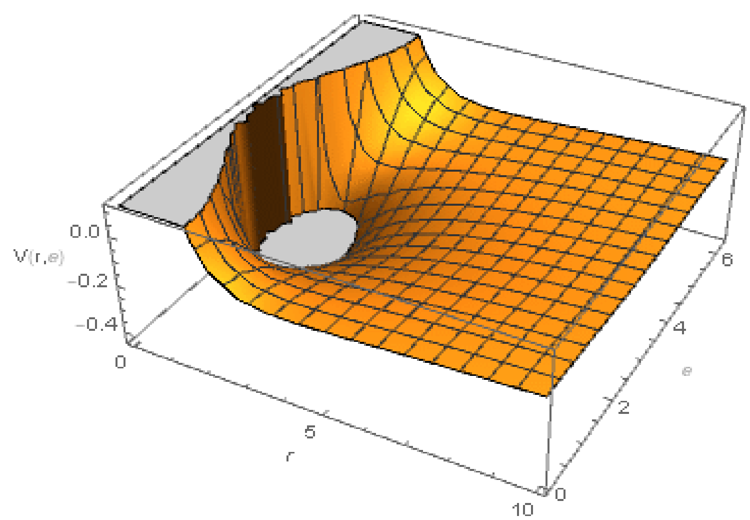

Figure 1.1: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos\theta)$ in terms of r and θ

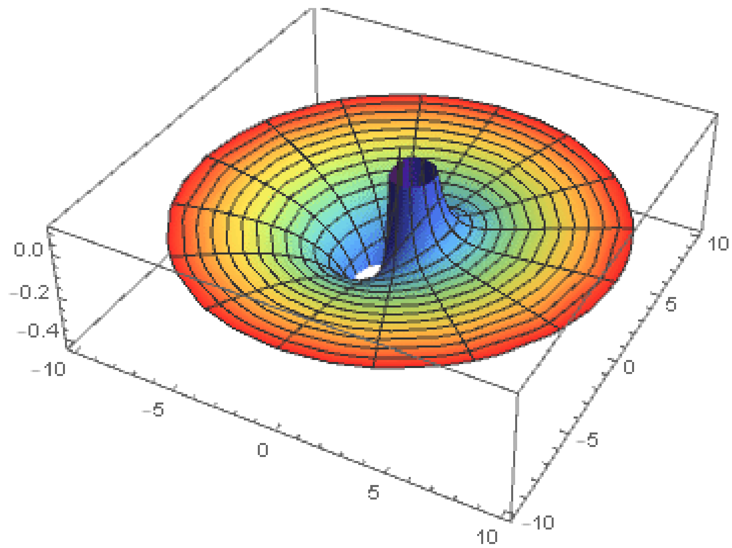

Figure 1.2: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos\theta)$ in terms of r and θ in cylindrical coordinates system

1.2 Non-Relativistic Studies of 2D Non-Central Potentials

1.2.1 2D Schrödinger Equation

To see the behavior of the potentials shown in the $Table1$, we plotted it in terms of r and θ,in cartisian coordinates system and in polar coordinates system the graphs shown in the figures [1.1, ..., 1.20] ,regarding the non-relativistic studies in this section we illustrated the solution of Schrödinger equation with the non-central potential of kind $V(r,\theta) = \mu\left[\frac{f(\theta)}{r^2} + V(r)\right]$,where μ is the mass ,$f(\theta)$ and $V(r)$ are mentioned in general introduction $Table1$.

The Schrödinger equation is written as

$$\left[\frac{-\hbar^2}{2\mu}\Delta + V(r,\theta)\right]\psi = E\psi \tag{1.1}$$

When we substitute the potential by its expression the Schrödinger equation of our system is

$$\left[\frac{-\hbar^2}{2\mu}\Delta + \mu\left(V(r) + \frac{f(\theta)}{r^2}\right)\right]\psi = E\psi \tag{1.2}$$

To solve this equation by the separation of variable method ,it is better the using of the polar coordinates (r,θ) ,in this case the Schrödinger equation is written as

$$\left[\frac{-\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) + \mu V(r) + \frac{\mu f(\theta)}{r^2}\right]\psi = E\psi \tag{1.3}$$

We put the equation in the more convenient following form:

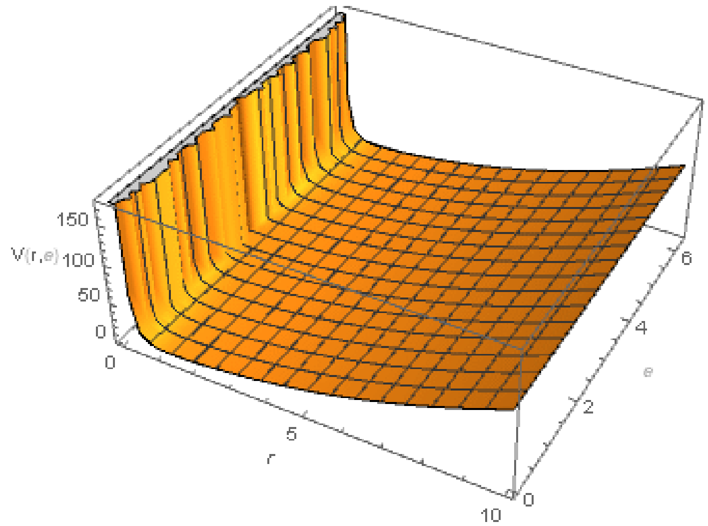

Figure 1.3: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos\theta)$ in terms of r and θ in

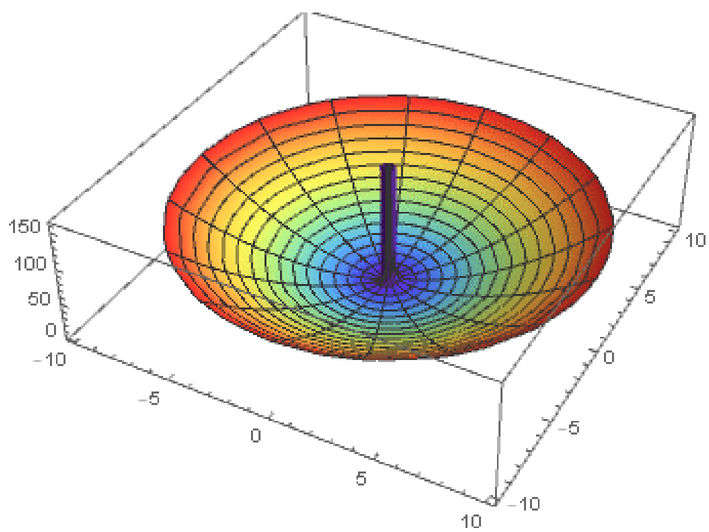

Figure 1.4: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos\theta)$ in terms of r and θ in cylindrical coordinates system

$$\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{2\mu^2}{\hbar^2}V(r)\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} - \frac{2\mu^2}{\hbar^2}\ f(\theta)\right)\right]\psi = -\frac{2\mu E}{\hbar^2}\psi \tag{1.4}$$

The variables can be separated when the wave function is written as :$\psi = r^{-\frac{1}{2}}R(r)\Theta(\theta)$, we have to calculate the derivatives of the wave function with the new form

The first derivative of ψ with respect to r in terms of the new form is

$$\frac{\partial\psi}{\partial r} = -\frac{1}{2}r^{-\frac{3}{2}}R(r)\Theta(\theta) + r^{-\frac{1}{2}}\frac{\partial R(r)}{\partial r}\Theta(\theta) \tag{1.5}$$

The second derivative of ψ with respect to r in terms of the new form is

$$\frac{\partial^2\psi}{\partial r^2} = \left[\frac{3}{4}r^{-\frac{5}{2}}R(r)\Theta(\theta) - r^{-\frac{3}{2}}\frac{\partial R(r)}{\partial r}\Theta(\theta) + r^{-\frac{1}{2}}\frac{\partial^2 R(r)}{\partial r^2}\Theta(\theta)\right] \tag{1.6}$$

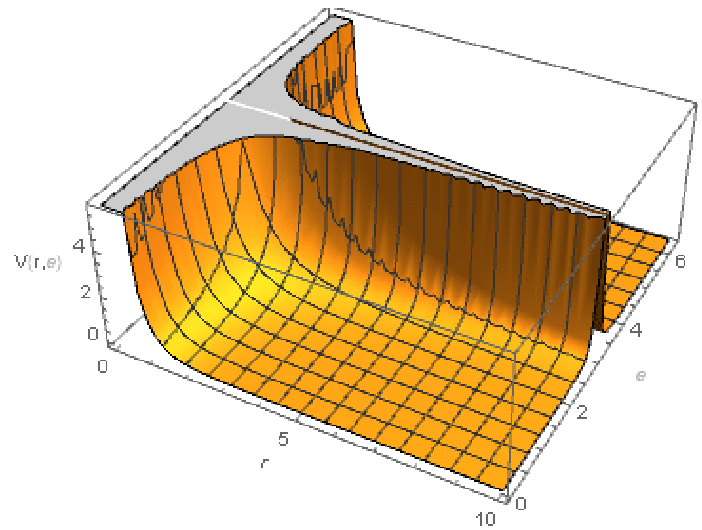

Figure 1.5: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\left(\alpha\sin^2\theta + \beta\sin\theta + \gamma\right)\cos^{-2}\right)$ in terms of r and θ

The second derivative of ψ with respect to θ is

$$\frac{\partial^2\psi}{\partial\theta^2} = r^{-\frac{1}{2}}R(r)\frac{\partial^2\Theta(\theta)}{\partial\theta^2} \tag{1.7}$$

We substitute the equations 1.5 ,1.6and 1.7 in the Schrödinger equation 1.4

$$\begin{aligned} r^{-\frac{1}{2}}\frac{\partial^2R(r)}{\partial r^2}\Theta(\theta) - r^{-\frac{3}{2}}\frac{\partial R(r)}{\partial r}\Theta(\theta) + \frac{3}{4}r^{-\frac{5}{2}}R(r)\Theta(\theta) - \frac{1}{2}r^{-\frac{5}{2}}R(r)\Theta(\theta) + \\ r^{-\frac{3}{2}}\frac{\partial R(r)}{\partial r}\Theta(\theta) - \frac{2\mu^2}{\hbar^2}V(r)r^{-\frac{1}{2}}R(r)\Theta(\theta) + \frac{2\mu E}{\hbar^2}r^{-\frac{1}{2}}R(r)\Theta(\theta) + \\ r^{-\frac{5}{2}}R(r)\frac{\partial^2\Theta(\theta)}{\partial\theta^2} - \frac{2\mu^2}{\hbar^2}\ f(\theta)r^{-\frac{5}{2}}R(r)\Theta(\theta) = 0 \end{aligned} \tag{1.8}$$

After some simplification we get the following equation:

$$\begin{aligned} r^{-\frac{1}{2}}\frac{\partial^2R(r)}{\partial r^2}\Theta(\theta) + \frac{1}{4}r^{-\frac{5}{2}}R(r)\Theta(\theta) - \frac{2\mu^2}{\hbar^2}V(r)r^{-\frac{1}{2}}R(r)\Theta(\theta) + \\ \frac{2\mu E}{\hbar^2}r^{-\frac{1}{2}}R(r)\Theta(\theta) + r^{-\frac{5}{2}}R(r)\frac{\partial^2\Theta(\theta)}{\partial\theta^2} - \frac{2\mu^2}{\hbar^2}\ f(\theta)r^{-\frac{5}{2}}R(r)\Theta(\theta) = 0 \end{aligned} \tag{1.9}$$

We divide the last equation by $r^{-\frac{5}{2}}$

$$\begin{aligned} \frac{1}{r^{-2}}\left[\frac{\partial^2R(r)}{\partial r^2}\Theta(\theta) + \frac{1}{4}r^{-2}R(r)\Theta(\theta) - \frac{2\mu^2}{\hbar^2}V(r)R(r)\Theta(\theta) + \frac{2\mu E}{\hbar^2}R(r)\Theta(\theta)\right] \\ = \left[-R(r)\frac{\partial^2\Theta(\theta)}{\partial\theta^2} + \frac{2\mu^2}{\hbar^2}\ f(\theta)R(r)\Theta(\theta)\right] \end{aligned} \tag{1.10}$$

To separated this equation we divide it by $R(r)\Theta(\theta)$ then we find the following equation

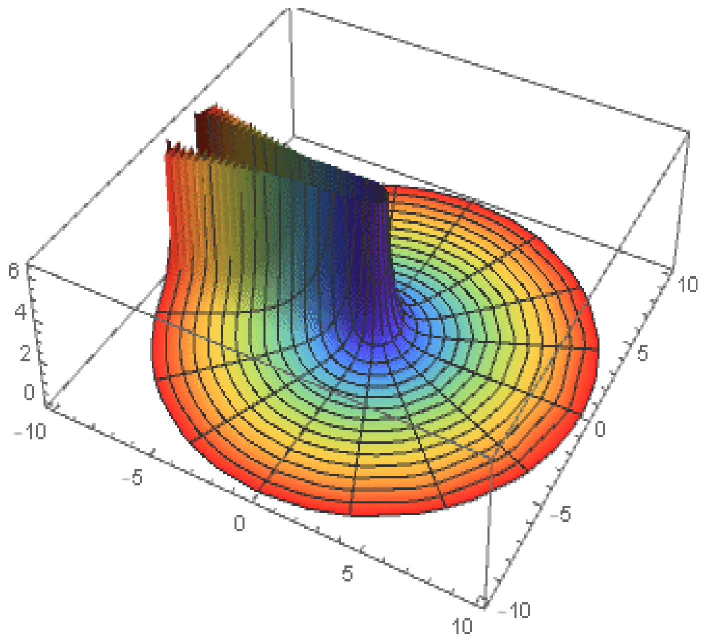

Figure 1.6: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\left(\alpha\sin^2\theta + \beta\sin\theta + \gamma\right)\cos^{-2}\right)$ in terms of r and θ in cylindrical coordinates system

$$\frac{1}{R(r)}\frac{1}{r^{-2}}\left[\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{4}r^{-2}R(r) - \frac{2\mu^2}{\hbar^2}V(r)R(r) + \frac{2\mu E}{\hbar^2}R(r)\right] = \frac{1}{\Theta(\theta)}\left[-\frac{\partial^2\Theta(\theta)}{\partial\theta^2} + \frac{2\mu^2}{\hbar^2}\ f(\theta)\Theta(\theta)\right] \tag{1.11}$$

When we put the both sides of equation 1.11equal to E_θ we find two equations as

$$\frac{1}{\Theta(\theta)}\left[-\frac{\partial^2\Theta(\theta)}{\partial\theta^2} + \frac{2\mu^2}{\hbar^2}\ f(\theta)\Theta(\theta)\right] = E_\theta \tag{1.12}$$

$$\frac{1}{R(r)}\frac{1}{r^{-2}}\left[\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{4}r^{-2}R(r) - \frac{2\mu^2}{\hbar^2}V(r)R(r) + \frac{2\mu E}{\hbar^2}R(r)\right] = E_\theta \tag{1.13}$$

Then it is easy to find the linear differential equations

$$\left[\frac{\partial^2\Theta(\theta)}{\partial\theta^2} - \frac{2\mu^2}{\hbar^2}\ f(\theta)\Theta(\theta)\right] = E_\theta\Theta(\theta) \tag{1.14}$$

$$\left[\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{4}r^{-2}R(r) - \frac{2\mu^2}{\hbar^2}V(r)R(r) + \frac{2\mu E}{\hbar^2}R(r)\right] = -r^{-2}E_\theta R(r) \tag{1.15}$$

So the equation 1.2 give us two equations

$$\frac{d^2\Theta(\theta)}{d\theta^2} - \left(E_\theta + \frac{2\mu^2}{\hbar^2}\ f(\theta)\right)\Theta(\theta) = 0 \tag{1.16}$$

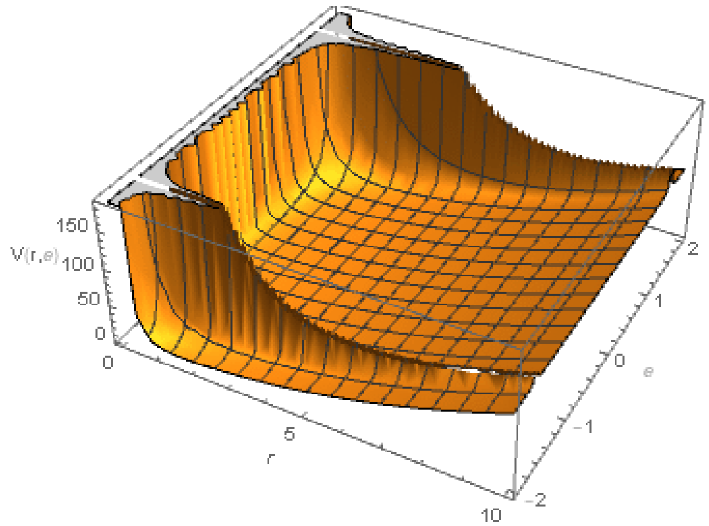

Figure 1.7: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\left(\alpha\sin^2\theta + \beta\sin\theta + \gamma\right)\cos^{-2}\right)$ in terms of r and θ

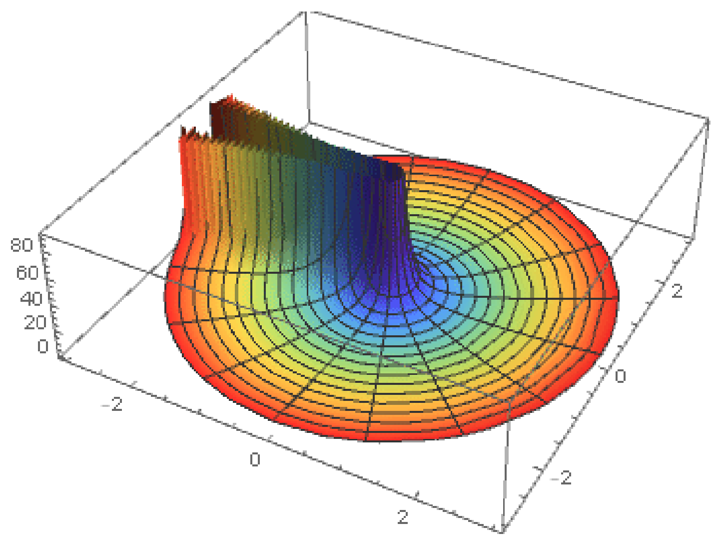

Figure 1.8: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\left(\alpha\sin^2\theta + \beta\sin\theta + \gamma\right)\cos^{-2}\right)$ in terms of r and θ in cylindrical coordinates system

$$\frac{d^2R(r)}{dr^2} + \left[\left(E_\theta + \frac{1}{4}\right)\frac{1}{r^2} - \frac{2\mu^2}{\hbar^2}V(r) + \frac{2\mu E}{\hbar^2}\right]R(r) = 0 \tag{1.17}$$

From the radial equation we can plot the effectiv potential $V_{eff} = -\left(E_\theta + \frac{1}{4}\right)\frac{1}{r^2} + \frac{2\mu^2}{\hbar^2}V(r)$ in terms of r and θ to show the existence of the bound state ,and now we have to solve the angular equation 1.16 to find the separation constant E_θ and then we substitute it in the solution of the radial equation 1.17; this will give us the energies E of the system and also the wave function $\psi\,(r,\theta)$.,where we take as applications the potentials of *Table* 1

1.2.2 Non-relativistic Energy and Wave Function (Applications)

in this subsection we calculated the energy and wave function of our system for previous potentials and will treat it case by case

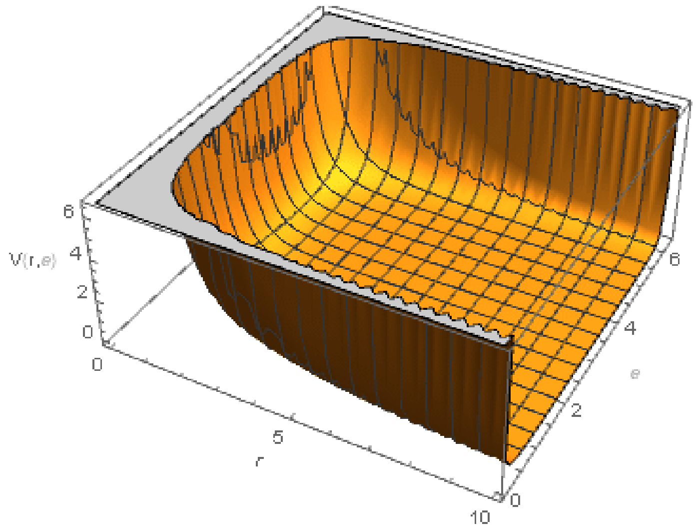

Figure 1.9: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\frac{\theta}{2} + \beta\tan\frac{\theta}{2} + \gamma\right)$ in terms of r and θ

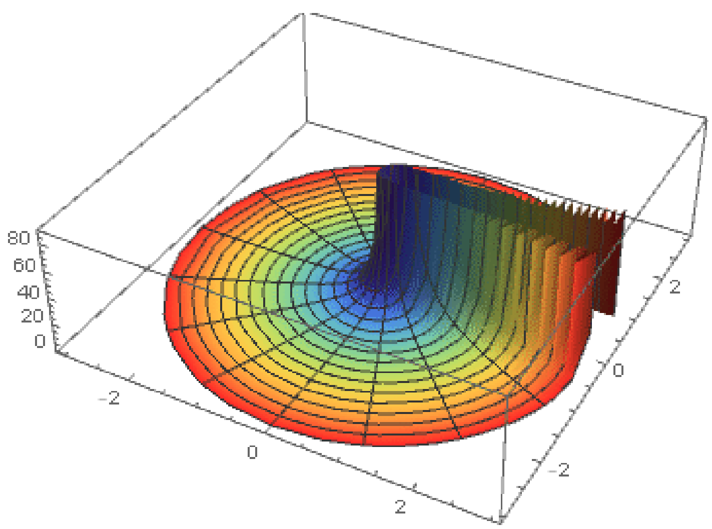

Figure 1.10: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\frac{\theta}{2} + \beta\tan\frac{\theta}{2} + \gamma\right)$ in terms of r and θ in cylindrical coordinates system

Case1:$V_1(r,\theta) = \mu\left[-\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos\theta)\right]$

The kratzer plus dipole potential we can find it in many chimecal and physical systems sach in ring-shaped organic molecules [33] [94] the Kratzer potential has been experimentally justified in 2D systems because Rydberg series of s-type excitonic states in monolayers of semiconducting transition metal dichalcogenides, [95] which are 2D semiconductors, follow a model system of 2D Kratzer type instead of a 2D hydrogen atom,in order to deduce the Kratzer potential of a system consisting of a point charge q under the effect of a non-zero distribution charge $Q = \int dq$ (a cluster of point charges dq). One this later create a Colombian potential in the space equal the sum of Colombian potential of elementary charge q_i can take as an example of this system a polar ion and a point charge. So the potential produced

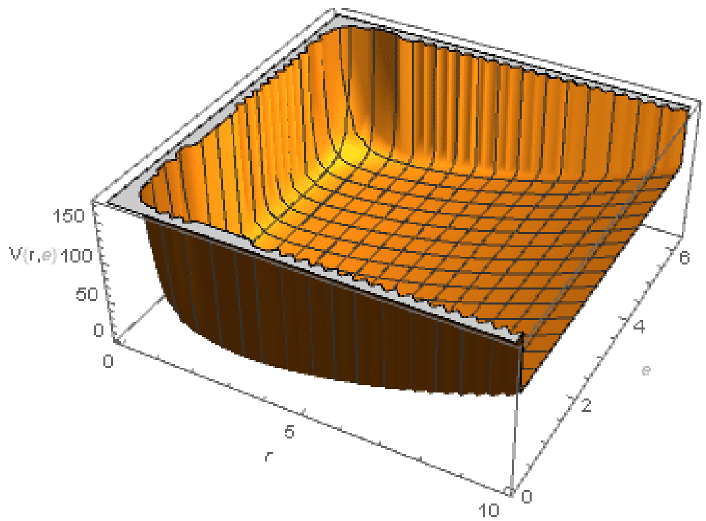

Figure 1.11: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\frac{\theta}{2} + \beta\tan\frac{\theta}{2} + \gamma\right)$ in terms of r and θ

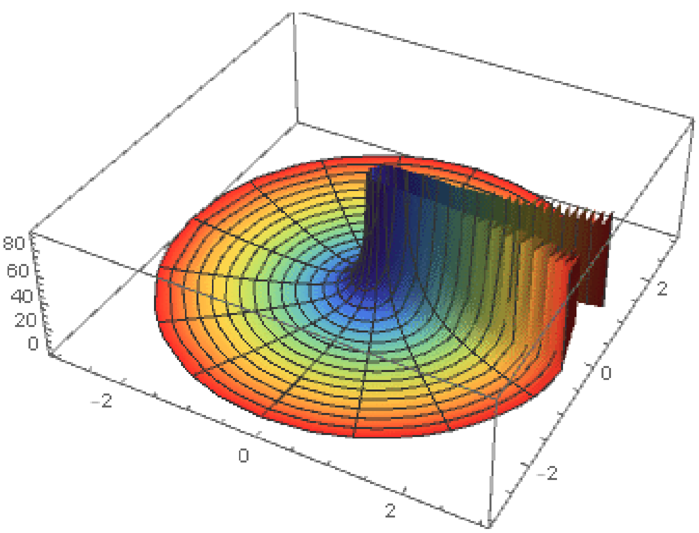

Figure 1.12: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\frac{\theta}{2} + \beta\tan\frac{\theta}{2} + \gamma\right)$ in terms of r and θ in cylindrical coordinates system

by the charge distribution at the position of the test charge q is written as follows

$$V(r) = \int \frac{1}{4\pi\varepsilon_0}\frac{dq_a}{r_a} \tag{1.18}$$

We choose a reference with the origin O be coincide with the center of the charge Q.,and we denoted M as the position of the test charge q,and it's vector position by the vector $\overrightarrow{r}$, the position of elementary charge relative to the test charged q is $\overrightarrow{r_a} = \overrightarrow{AM} = \overrightarrow{AO} - \overrightarrow{OM} = \overrightarrow{r} - \overrightarrow{a}$ (when the position of the charged dq_adenoted by $A$and defined by the vector $\overrightarrow{a}$. Thus we write

$$V(r) = \int \frac{1}{4\pi\varepsilon_0}\frac{dq_a}{\left\|\overrightarrow{r} - \overrightarrow{a}\right\|} = \int \frac{1}{4\pi\varepsilon_0}dq_a\left[\left(\overrightarrow{r} - \overrightarrow{a}\right)^2\right]^{-\frac{1}{2}} \tag{1.19}$$

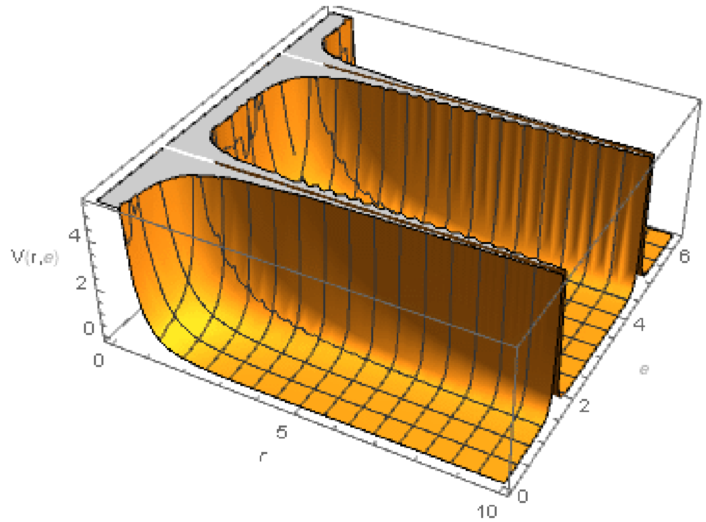

Figure 1.13: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cot^2\frac{\theta}{2} + \beta\cot\frac{\theta}{2} + \gamma\right)$ in terms of r and θ

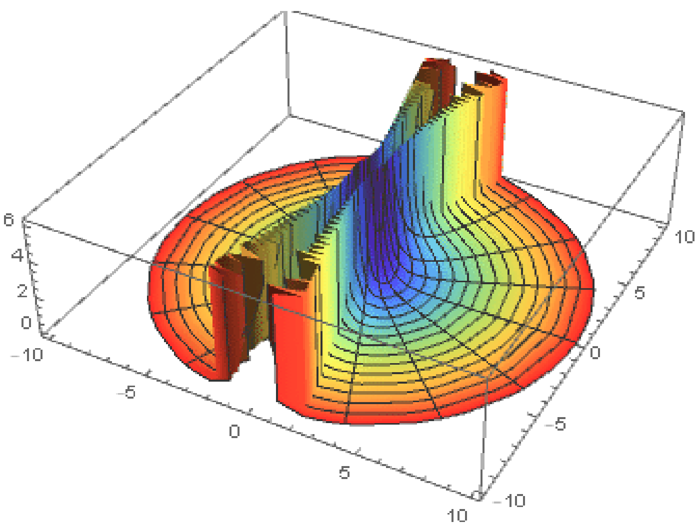

Figure 1.14: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cot^2\frac{\theta}{2} + \beta\cot\frac{\theta}{2} + \gamma\right)$ in terms of r and θ in cylindrical coordinates system

By spreading the square ,we find

$$V(r) = \int \frac{1}{4\pi\varepsilon_0} dq_a \left[\left(\overrightarrow{r}^2 - 2\overrightarrow{r}.\overrightarrow{a} + \overrightarrow{a}^2\right)\right]^{-\frac{1}{2}} \tag{1.20}$$

To finding the Colombian we extract $\dfrac{1}{r}$ so

$$V(r) = \int \frac{1}{4\pi\varepsilon_0}\frac{dq_a}{r}\left[\left(1 - 2\frac{\overrightarrow{r}.\overrightarrow{a}}{\overrightarrow{r}^2} + \frac{\overrightarrow{a}^2}{\overrightarrow{r}^2}\right)^2\right]^{-\frac{1}{2}} \tag{1.21}$$

We suppose that the dimensions of the extended charge Q are small compared to those of the whole system constituted by Q and the point charge q, such that we write $|a| \prec\prec |r|$, and we use the Taylor series thus we have

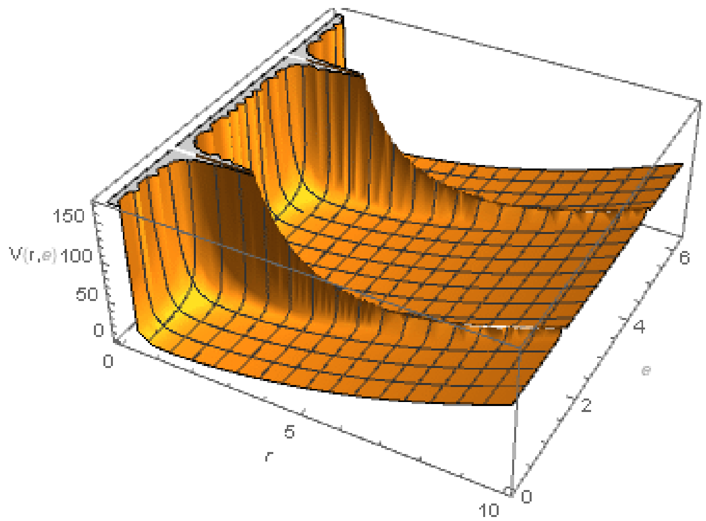

Figure 1.15: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cot^2\frac{\theta}{2} + \beta\cot\frac{\theta}{2} + \gamma\right)$ in terms of r and θ

$$\left[\left(1 - 2\frac{\overrightarrow{r}.\overrightarrow{a}}{\overrightarrow{r}^2} + \frac{\overrightarrow{a}^2}{\overrightarrow{r}^2}\right)^2\right]^{-\frac{1}{2}} = 1 + \frac{\overrightarrow{r}.\overrightarrow{a}}{r^2} + O\left(\frac{\overrightarrow{a}^2}{\overrightarrow{r}^2}\right) \tag{1.22}$$

We restrict ourselves to the $1st$ order of the multipole expansion

$$V(r) = \int \frac{1}{4\pi\varepsilon_0}\frac{dq_a}{r}\left[1 + \frac{\overrightarrow{r}.\overrightarrow{a}}{r^2}\right] \Longrightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\left(\int\frac{dq_a}{r} + \int\frac{dq_a}{r}\frac{\overrightarrow{r}.\overrightarrow{a}}{r^2}\right) \tag{1.23}$$

By substitute the result of the scalar product we find

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r}\int_0^Q dq + \int_0^Q \frac{a\cos\theta_a\ dq_a}{r^2}\right) \tag{1.24}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} + \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int a\cos\theta_a dq_a \tag{1.25}$$

we put $\int a\cos\theta_a dq_a = D_r$ and $\frac{1}{4\pi\varepsilon_0}D_r = ad_r$,d_r is the dissociation energy and a is the equilibrium internuclear separation,thus the Kratzer potential is writing as follow

$$V\ (r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} + \frac{1}{4\pi\varepsilon_0}\frac{D_r}{r^2} \tag{1.26}$$

We see that the potential is central and this may not reflect reality because the distribution is not usually perfectly symmetric. Therefore, we have to take into account the possible anisotropy

in the charge distribution and to do this we consider that the positive and the negative centers of charges do not coincide in Q and we denote their positions $\overrightarrow{a_+}$ and $\overrightarrow{a_-}$ This two centers form an electric dipole representing this anisotropy and the potential of such a dipole is just $\frac{D_\theta\cos\theta}{r^2}$.The dipole moment D_θ is proportional to the distance between the two charge

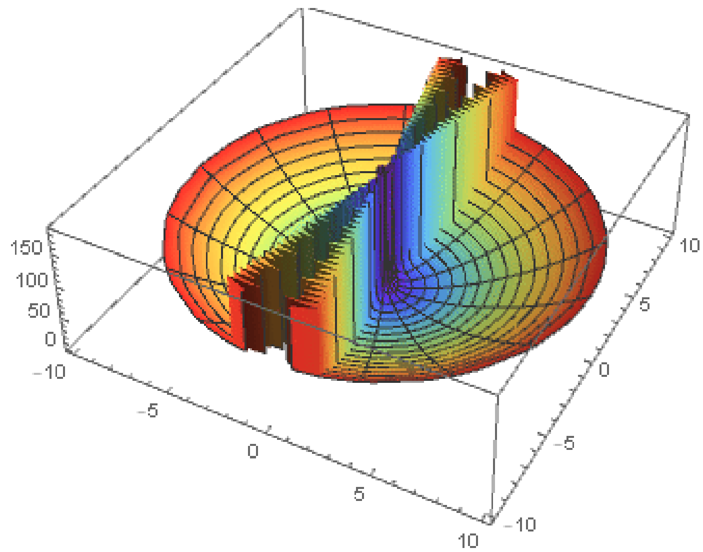

Figure 1.16: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cot^2\frac{\theta}{2} + \beta\cot\frac{\theta}{2} + \gamma\right)$ in terms of r and θ in cylindrical coordinates system

centers and the angle θ defines the orientation of the position $\overrightarrow{r}$ according to the dipole axis defined by $\overrightarrow{a_+}\,\overrightarrow{a_-}$. We call this term the "angular" dipole to differentiate it from the "radial" one. adding all the terms together gives us the Coulomb potential with two dipoles and we call it a non-central (N-C) Kratzer potential i

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} + \frac{1}{4\pi\varepsilon_0}\frac{D_r}{r^2} + \frac{1}{4\pi\varepsilon_0}\frac{D_\theta\cos\theta}{r^2} \tag{1.27}$$

To keep the labels of Hauto we put $\frac{qQ}{4\pi\varepsilon_0} = -\mu H$, $\frac{qD_r}{4\pi\varepsilon_0} = \mu D_r$ and $\frac{qD_\theta}{4\pi\varepsilon_0} = \frac{\hbar^2}{2\mu}\alpha$ where the dipole potential take the forme $f(\theta) = \frac{\hbar^2}{2\mu^2}\alpha\cos(\theta)$,we substitute it in the angular equation 1.16 to becomes

$$\left(\frac{d^2}{d\theta^2} - E_\theta - \alpha\cos(\theta)\right)\Theta(\theta) = 0 \tag{1.28}$$

We put the following changes to get a known equation

$$\theta = 2z \quad (\theta\ is\ 2\pi\ periodic\ and\ z\ is\ \pi\ periodic) \tag{1.29}$$

And

$$a = -4E_\theta, p = 2\alpha \tag{1.30}$$

So when we substitute the new parameters the angular equation 1.28 becomes

$$\frac{d^2\Theta(z)}{dz^2} + (a - 2p\cos(2z))\Theta(Z) = 0 \tag{1.31}$$

This equation is Mathieu equation [37].and its solutions are the cosine-elliptic $ce_{2m}(z)$ and the sine-elliptic $se_{2m+2}(z)$ functions where m is a natural number [38] . The solutions of the Mathieu equation are periodic because z has π as a period and this leads us to

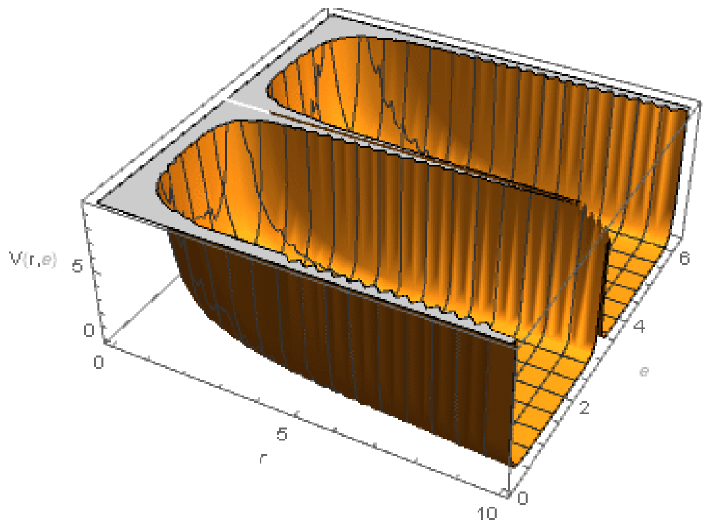

Figure 1.17: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\tan^2\theta + \beta\tan\theta + \gamma)$ in terms of r and θ

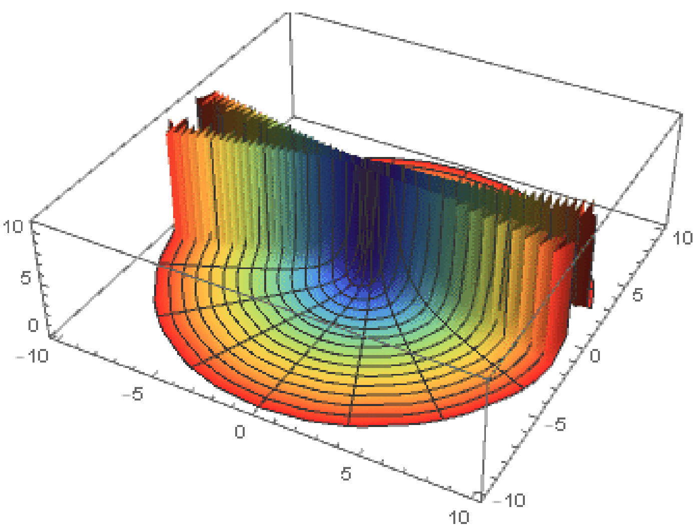

Figure 1.18: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\tan^2\theta + \beta\tan\theta + \gamma)$ in terms of r and θ in cylindrical coordinates system

consider the Floquet's theorem [39] or the Bloch's theorem [40]. They stipulate that, for a given value of the parameter p, the solution is periodic only for certain values of the other parameter a; They are called characteristic values and denoted $a(2m,p)$ or $a_{2m}(p)$ for the *ce* solutions and $b(2m,p)$ or $b_{2m}(p)$ for the *se* ones..There is no analytical expression for the Mathieu characteristic values $a_{2m}(p)$ and $b_{2m}(p)$, so they are usually given either numerically or graphically. This doesn't preclude that we can write approximate analytical expressions for small and large values of α [41].

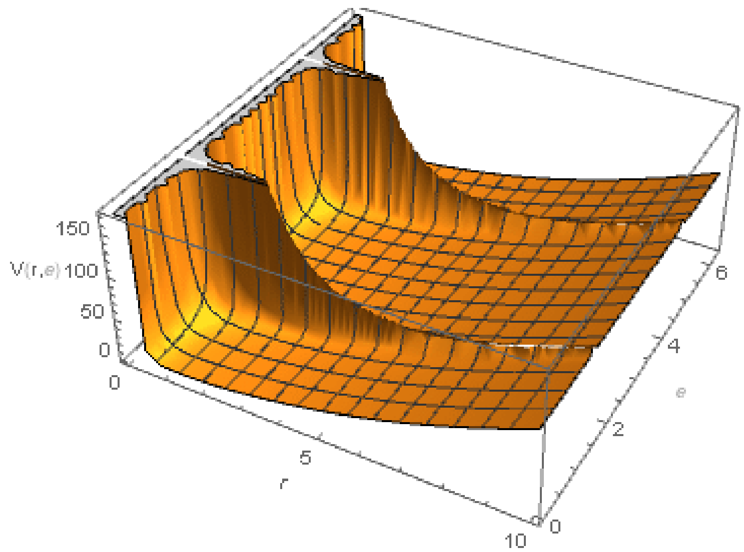

Figure 1.19: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\tan^2\theta + \beta\tan\theta + \gamma)$ in terms of r and θ

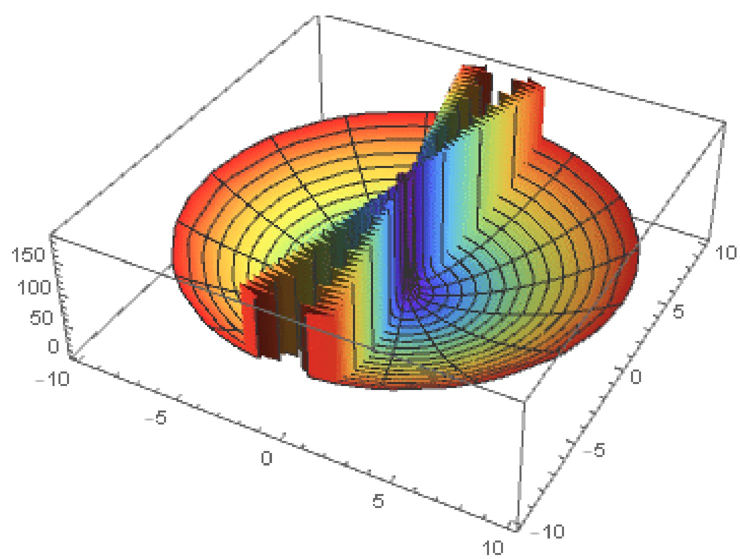

Figure 1.20: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\tan^2\theta + \beta\tan\theta + \gamma)$ in terms of r and θ in cylindrical coordinates system

For small values of α, we can express a and b for $m > 3$ as $(l = 4m2 - 1)$:

$$a_{(2m)}(p) = b_{(2m)}(p) = 4m^2 + \frac{1}{2(4m^2-1)}p^2 + \frac{20m^2+7}{32(4m^2-1)^3(4m^2-4)}p^4 + \frac{36m^4+232m^2+29}{64(4m^2-1)^5(4m^2-4)(4m^2-9)}p^6 + O(p^8) \tag{1.32}$$

The coefficients of the power series of $a_{2m}(p)$ and $b_{2m}(p)$ are the same until the terms in p^{2m-2} ,we have similar polynomials for $m \leq 3$ but with different coefficients for the a's and the b's..We note here that there is no se solutions for $m = 0$ and so there is no $b(m = 0)$.

For large values of p, we get another polynomial $(k = 2n + 1)$:

$$a_n(p) = b_{n+1}(p) = -2p + 2kp^{1/2} - \frac{1}{8}\left[k^2+1\right] - \left[k^3+3k\right]\frac{1}{2^7p^{1/2}} - \left[5k^4+34k^2+9\right]\frac{1}{2^{12}p} + O\left(p^{-3/2}\right) \tag{1.33}$$

From now we use the same symbol $c_{2m}(p)$ for both characteristic values $a_{2m}(p)$ and $b_{2m}(p)$. Using the equation 1.30 ,we get energy as:

$$E_\theta = E_\theta^{2m} = -\frac{1}{4}a = -\frac{1}{4}c_{2m}(p) \tag{1.34}$$

And the angular wave function is Mathieu function $\Theta(\theta)$

$$\Theta(\theta) = Mathieufunction \tag{1.35}$$

From 1.32, we see that for small values of α (or p), the angular solution can be put in the form:

$$E_\theta^{(2m)} = -m^2 + P_m(\alpha) \tag{1.36}$$

Where$P_m(\alpha)$ is a polynomial in terms of even power of α starting from 2. This expression will be used to validate our solutions in the limit $\alpha \to 0$ (or $p \to 0$). In this case, we see from 1.32 that $a_{2m}(p)$ and $b_{2m}(p)$ have the same limit $4m^2$. So the $ce_{2m}(z)$and $se_{2m+2}(z)$ are degenerate and the solution becomes a linear combination of $\cos(2mz)$, which is the limit of $ce_{2m}(z)$, and $\sin(2mz)$, which is the limit of $se_{2m+2}(z)$; Here we retrieve the solution $exp(2imz)$of 1.31 for $p = 0$.

We substitute by the kratzer plus dipole potential ,the radial equation 1.17 of this case is

$$\frac{d^2R(r)}{dr^2} + \left[\left(E_\theta + \frac{1}{4} - \frac{2\mu^2 D_r}{\hbar^2}\right)\frac{1}{r^2} + \frac{2\mu^2}{\hbar^2}\frac{H}{r} + \frac{2\mu E}{\hbar^2}\right]R(r) = 0 \tag{1.37}$$

To solve the equation 1.37 we use the following change

$$R(r) = r^\lambda e^{-\beta r} f(r) \tag{1.38}$$

The first derivative of R in terms of new functions is

$$\frac{dR}{dr} == \frac{d\left(r^{\lambda}e^{-\beta r}f(r)\right)}{dr} = \lambda r^{\lambda-1}e^{-\beta r}f(r) - \beta r^{\lambda}e^{-\beta r}f(r) + r^{\lambda}e^{-\beta r}\frac{df(r)}{dr} \tag{1.39}$$

The second derivative of R in terms of new functions is

$$\begin{aligned}\frac{d^2R}{dr^2} &= r^{\lambda}e^{-\beta r}\frac{d^2f(r)}{dr^2} + \left(2\lambda r^{\lambda-1}e^{-\beta r} - 2\beta r^{\lambda}e^{-\beta r}\right)\frac{df(r)}{dr}\\ &+ \left(\lambda(\lambda-1)r^{\lambda-2}e^{-\beta r} - 2\beta\lambda r^{\lambda-1}e^{-\beta r} + \beta^2 r^{\lambda}e^{-\beta r}\right)f(r)\end{aligned} \tag{1.40}$$

We substitute the results of 1.40 in the equation 1.37 we find

$$\begin{aligned}& r^{\lambda}e^{-\beta r}\frac{d^2f(r)}{dr^2} + \left(2\lambda r^{\lambda-1}e^{-\beta r} - 2\beta r^{\lambda}e^{-\beta r}\right)\frac{df(r)}{dr}\\ &+ \left(\lambda(\lambda-1)r^{\lambda-2}e^{-\beta r} - 2\beta\lambda r^{\lambda-1}e^{-\beta r} + \beta^2 r^{\lambda}e^{-\beta r}\right)f(r)\\ &+ \left[\left(E_\theta + \frac{1}{4} - \frac{2\mu^2 D_r}{\hbar^2}\right)\frac{1}{r^2} + \frac{2\mu^2}{\hbar^2}\frac{H}{r} + \frac{2\mu E}{\hbar^2}\right]r^{\lambda}e^{-\beta r}f(r) = 0\end{aligned} \tag{1.41}$$

We divide by $r^{\lambda-1}e^{-\beta r}$,we get

$$\begin{aligned}& r\frac{d^2f(r)}{dr^2} + (2\lambda - 2\beta r)\frac{df(r)}{dr}\\ &+ \left(\lambda(\lambda-1)r^{-1} - 2\beta\lambda + \beta^2 r\right)f(r)\\ &+ \left[\left(E_\theta + \frac{1}{4} - \frac{2\mu^2 D_r}{\hbar^2}\right)\frac{1}{r^2} + \frac{2\mu^2}{\hbar^2}\frac{H}{r} + \frac{2\mu E}{\hbar^2}\right]rf(r) = 0\end{aligned} \tag{1.42}$$

After some simplification we find this equation

$$\begin{aligned}& r\frac{d^2f(r)}{dr^2} + 2(\lambda - \beta r)\frac{df(r)}{dr}\\ &+ \left[\left(\lambda(\lambda-1) + E_\theta - \frac{2\mu^2 D_r}{\hbar^2} + \frac{1}{4}\right)\frac{1}{r} + \left(\frac{2\mu^2}{\hbar^2}H - 2\beta\lambda\right) + \left(\frac{2\mu E}{\hbar^2} + \beta^2\right)r\right]f(r) = 0\end{aligned} \tag{1.43}$$

Because the parametersβ and λ are free ones, we chose them as follows to simplify the equation:

$$\beta^2 = -\frac{2\mu E}{\hbar^2} \tag{1.44}$$

$$\lambda(\lambda-1) + E_\theta - \frac{2\mu^2 D_r}{\hbar^2} + \frac{1}{4} = 0 \tag{1.45}$$

So we get a new differential equation for $f(r)$ as

$$r\frac{d^2f(r)}{dr^2} + 2(\lambda - \beta r)\frac{df(r)}{dr} - 2\left(\frac{\mu^2 H}{\hbar^2} + \lambda\beta\right)f(r) = 0 \tag{1.46}$$

As $\psi(r,\theta)$ must be convergent, the accepted solutions for these parameters that let $R(r)$

nonssingular at $r = 0$ are:

$$\beta = \sqrt{-\frac{2\mu E}{\hbar^2}} \tag{1.47}$$

And

$$\lambda = \frac{1}{2} + \sqrt{-E_\theta^{(m)} + \frac{2\mu^2 D_r}{\hbar^2}} \tag{1.48}$$

To change the equation to a known equation we define a new variable :

$$z = 2\beta r \Longrightarrow r = \frac{1}{2\beta} z \tag{1.49}$$

And

$$\frac{dz}{dr} = 2\beta \tag{1.50}$$

We calculate the derivatives of a function $f(r)$ in terms of the derivatives with respect to the new variable z

$$\frac{df(r)}{dr} = \frac{df(r)}{dz}\frac{dz}{dr} = 2\beta\frac{df(r)}{dz} \tag{1.51}$$

And the second derivatives is

$$\frac{d^2 f(r)}{dr^2} = 4\beta^2 \frac{d^2 f(r)}{dz^2} \tag{1.52}$$

We substitute this derivatives in the equation 1.46 we get

$$2\beta z \frac{d^2 f(z)}{dz^2} + 2(\lambda - \frac{1}{2}z)2\beta\frac{df(z)}{dz} - 2(\frac{\mu^2 H}{\hbar^2} + \lambda\beta)f(z) = 0 \tag{1.53}$$

We divide by 2β we get a confluent hypergeometric :

$$z\frac{d^2 f(z)}{dz^2} + (2\lambda - z)\frac{df(z)}{dz} - (\frac{\mu^2 H}{\hbar^2}\frac{1}{\beta} + \lambda)f(z) = 0 \tag{1.54}$$

The solution here is just the confluent hypergeometric function:[36]

$$f(z) = N_1 F_1\left(\lambda + \frac{\mu^2 H}{\hbar^2}\beta^{-1}, 2\lambda, z\right) \tag{1.55}$$

And

$$f(r) = N_1 F_1\left(\lambda + \frac{\mu^2 H}{\hbar^2}\beta^{-1}, 2\lambda, 2\beta r\right) \tag{1.56}$$

${}_1F_1\left(\lambda + \frac{\mu H}{\hbar^2}\beta^{-1}, 2\lambda, 2\beta r\right)$ can be written as Laguerre polynomials of degree n_r

$$L_{n_r}^{2\lambda-1}(2\beta r) = \frac{(n_r + 2\lambda - 1)!}{n_r!(2\lambda - 1)!}\, {}_1F_1\left(\lambda + \frac{\mu^2 H}{\hbar^2}\beta^{-1}, 2\lambda, 2\beta r\right) \tag{1.57}$$

To find the radial wave function of the system$R(r)$,we use the equation 1.38,so

$$R(r) = N_r r^{\lambda} e^{-\beta r} \, {}_1F_1\left(\lambda + \frac{\mu^2 H}{\hbar^2}\beta^{-1}, 2\lambda, 2\beta r\right) \tag{1.58}$$

We substitute by $\lambda = \frac{1}{2} + \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}$and $\beta^2 = -\frac{2\mu E}{\hbar^2}$

$$R(r) = N_r r^{\frac{1}{2}+\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}} e_1^{-\sqrt{-\frac{2\mu E}{\hbar^2}} r}$$
$${}_1F_1\left(\frac{1}{2} + \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}} + \frac{\mu^2 H}{\hbar}\sqrt{-\frac{1}{2\mu E}}, 1 + 2\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}, 2\sqrt{-\frac{2\mu E}{\hbar^2}} r\right) \tag{1.59}$$

N_r is a normalization constant

From the asymptotic behavior of the confluent series ($r \to \infty \Longrightarrow {}_1F_1 = 0$) which lead to $\psi \to 0$ when $r \to \infty$ we find the general condition of quantization :

$$\lambda + \frac{\mu^2 H}{\hbar^2}\beta^{-1} = -n_r, n_r = 0, 1, 2, ... \tag{1.60}$$

We use the relation$\lambda = \frac{1}{2} + \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}$, $\lambda + \frac{\mu^2 H}{\hbar^2}\beta^{-1} = -n_r$ and $\beta^2 = -\frac{2\mu E}{\hbar^2}$ to obtain the energy of our system

$$\lambda + \frac{\mu^2 H}{\hbar^2}\beta^{-1} = -n_r \Longrightarrow \beta^2 = \left(\frac{\hbar^2}{\mu^2 H}\right)^{-2} (n_r + \lambda)^{-2} \tag{1.61}$$

And

$$\beta^2 = -\frac{2\mu E}{\hbar^2} \Longrightarrow E = -\frac{\hbar^2}{2\mu}\beta^2 \tag{1.62}$$

So

$$E = -\frac{\hbar^2}{2\mu}\left(\frac{\hbar^2}{\mu^2 H}\right)^{-2} (n_r + \lambda)^{-2} \tag{1.63}$$

We substitute by the expression of $\lambda = \frac{1}{2} + \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}$we find the radial energy in terms of angular energy as

$$E_{n_r} = -2\frac{\mu^3 H^2}{\hbar^2}\left(2n_r + 2\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}} + 1\right)^{-2} \tag{1.64}$$

$n_r = 0, 1, 2, ...$

The radial equation is the same of the potential of case1 thus the energy expression and the radial part of the wave function is the same then we substitute the constant of separation1.34in energy expression 1.64 ,we find the final expression energy of the system as

$$E_{1(n_r,m)} = -2\frac{\mu^3 H^2}{\hbar^2}\left(2n_r + 2\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{2\mu^2 D_r}{\hbar^2}} + 1\right)^{-2} \tag{1.65}$$

$c_{2m}\left(2\alpha\right)$ is characteristic values of Mathieu function

$$\begin{aligned} a_{(2m)}\left(2\alpha\right) = b_{(2m)}\left(2\alpha\right) = 4m^2 + \frac{1}{2\left(4m^2-1\right)}\left(2\alpha\right)^2 + \frac{20m^2+7}{32\left(4m^2-1\right)^3\left(4m^2-4\right)}\left(2\alpha\right)^4 \\ + \frac{36m^4+232m^2+29}{64\left(4m^2-1\right)^5\left(4m^2-4\right)\left(4m^2-9\right)}\left(2\alpha\right)^6 + O\left(\left(2\alpha\right)^8\right) \end{aligned} \tag{1.66}$$

$n_r = 0, 1, 2, ...$,and $m = 0, 1, 2, ...$

We deduce the wave function of our system $\psi\left(r,\theta\right) = r^{-\frac{1}{2}}R(r)\Theta(\theta)$ from the angular part 1.35 and the radial part 1.59

$$\psi_1 = Nr^{\lambda-\frac{1}{2}}e^{-\beta r}\Theta\left(\theta\right)_1 F_1\left(\lambda + \frac{\mu H}{\hbar^2}\beta^{-1}, 2\lambda, 2\beta r\right) \tag{1.67}$$

Where $\beta = \sqrt{-\frac{2mE}{\hbar^2}}$and $\lambda = \frac{1}{2} + \sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{2\mu^2 D_r}{\hbar^2}}$

For the potential $\boldsymbol{V}_2\left(r,\theta\right) = \frac{\mu}{q}\left[-\frac{H}{r} + \left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_1\left(r,\theta\right)$ when we put $D_r \longrightarrow 0$ so

$$E_{2(n_r,m)} = -2\frac{\mu^3 H^2}{\hbar^2}\left(2n_r + 2\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)} + 1\right)^{-2} \tag{1.68}$$

The wave function is

$$\psi_2 = Nr^{\lambda-\frac{1}{2}}e^{-\beta r}\Theta\left(\theta\right)\ {}_1F_1\left(\lambda + \frac{\mu H}{\hbar^2}\beta^{-1}, 2\lambda, 2\beta r\right) \tag{1.69}$$

Where $\beta = \sqrt{-\frac{2mE}{\hbar^2}}$and $\lambda = \frac{1}{2} + \sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)}$

The energy of charged particles moving in a non pure dipole potential and under the effect of Kratzer potential as we obtained it is$\frac{qQ}{4\pi\varepsilon_0} = -\mu H$, $\frac{qD_r}{4\pi\varepsilon_0} = \mu D_r$ and $\frac{qD_\theta}{4\pi\varepsilon_0} = \frac{\hbar^2}{2\mu}\alpha$

$$E_{1(n_r,m)} = -\left[\left(\frac{4\pi\varepsilon_0\hbar^2}{\mu qQ}\sqrt{\frac{2\mu}{\hbar^2}}\right)\left(n_r + \sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{2\mu q D_r}{4\pi\varepsilon_0\hbar^2}} + \frac{1}{2}\right)\right]^{-2} \tag{1.70}$$

Starting from this expression, we can get the solutions of the usual $2D$ Kratzer potential [96],[97] by taking $H = -\frac{qQ}{4\pi\varepsilon_0\mu}$,$D_r = \frac{qD_r}{4\pi\varepsilon_0\mu}$,and $\alpha = \frac{\mu q D_\theta}{2\pi\varepsilon_0\hbar^2}$,the limit $\alpha \rightarrow 0$,and $P_m(2\alpha) \rightarrow 0$ which lead to $E_\theta^{(m)} = \frac{1}{4}c_{2m}\left(2\alpha\right) \rightarrow -m^2$, so

$$E_{n_r,m} = -\left[\left(\frac{4\pi\varepsilon_0\hbar^2}{\mu qQ}\sqrt{\frac{2\mu}{\hbar^2}}\right)\left(n_r + \sqrt{m^2 + \frac{2\mu q D_r}{4\pi\varepsilon_0\hbar^2}} + \frac{1}{2}\right)\right]^{-2} \tag{1.71}$$

Make the liaison with the Coulomb energies when $D_r \to 0$ so

$$E_{n_r,m} = -\left(\frac{4\pi\varepsilon_0\hbar^2}{\mu q Q}\sqrt{\frac{2\mu}{\hbar^2}}\right)\left(n_r + |m| + \frac{1}{2}\right)^{-2} = -\left(\frac{4\pi\varepsilon_0\hbar^2}{\mu q Q}\sqrt{\frac{2\mu}{\hbar^2}}\right)\left(n + \frac{1}{2}\right)^{-2} \tag{1.72}$$

this is the energy of the electron in the 2D hydrogen atom [98],[99] .So we obtain $n = n_r + |m|$ and $n_r = n - |m|$, now the energy eigenvalues of the system is written as

$$E_{n,m} = -\left[\left(\frac{4\pi\varepsilon_0\hbar^2}{\mu q Q}\sqrt{\frac{2\mu}{\hbar^2}}\right)\left(n - |m| + \sqrt{\frac{1}{4}c_{2m}(2\alpha) + \frac{2mqD_r}{4\pi\varepsilon_0\hbar^2}} + \frac{1}{2}\right)\right]^{-2} \tag{1.73}$$

Where n is the principal quantum number and m is the angular quantum number

For our numerical computations, we use the same considerations as those of molecular systems. We choose the extended charge as a positive ion and the point charge is an electron, so we get two opposite charges equal in magnitude $q = -Q = -e$. We use the Hartree atomic units where $\hbar = e = \mu = 4\pi\varepsilon_0 = 1$ and $\alpha = 2D_\theta$ the energies become:

$$E_{n,m} = -\left(n - |m| + \sqrt{\frac{1}{4}c_{2m}(4D_\theta) + 2D_r} + \frac{1}{2}\right)^{-2} \tag{1.74}$$

We note in this relation of the energies, that the angular dipole removes the degeneracy of the se and ce states for $m \neq 0$ This degeneracy is restored when the angular moment vanishes since the two Mathieu's characteristic parameters a_{2m} and b_{2m} have the same limit in this case equation 1.32. The result restores those of the ordinary Kratzer potential (or Coulomb potential) where the wave function of each level $E_{n,m}$ is a linear combination of bothse and ce states. For the s-states ($m = 0$), we only find the ce solutions because the se solutions are absent in this case. Through the expression 1.74, we see that the behavior of the energies follows essentially that of the Mathieu's parameters and thus the angular moment, whereas the effect of the radial moment merely shifts the energies to larger or smaller values according to its sign. The sign of the angular moment doesn't affect the results because the parameters c_{2m} are even functions. ,of course, the energies increase with the n and decrease with the m but the main effect of the m is to extend the allowed region for the values of the angular momentum. We also note that the energies corresponding to the $ce_{2m}(z)$ solutions (we note them $E^a_{n,m}$) are larger than the $se_{2m}(z)$ ones (noted $E^b_{n,m}$) and this is caused by the fact that the a_{2m} are bigger than the b_{2m} see($Figures 1.21, 1.22, 1.23$ *and* 1.24). The main remark that can be drawn from 1.74 is that there is an essential condition for the system to have bound states:

The condition is that $E_{n_r,m}$ is real this means

$$2D_r + \frac{1}{4}c_{2m}(4D_\theta) \succeq 0 \tag{1.75}$$

This condition shows that there are critical values for the two dipole moments, depending

D_r	m	0	1	2	3
-0.3	ce	–	1.925	8.004	17.462
	se	–	–	4.553	12.308
0	ce	–	2.662	8.679	18.132
	se	–	1.947	5.241	12.976
0.3	ce	0.543	3.284	9.323	18.782
	se	–	1.526	5.878	13.624
0.6	ce	0.923	3.851	9.942	19.420
	se	–	2.027	6.479	14.254
0.9	ce	12.84	4.385	10.543	20.046
	se	–	2.291	7.054	14.870

Table 1.1: Critical values for the dipole momentum

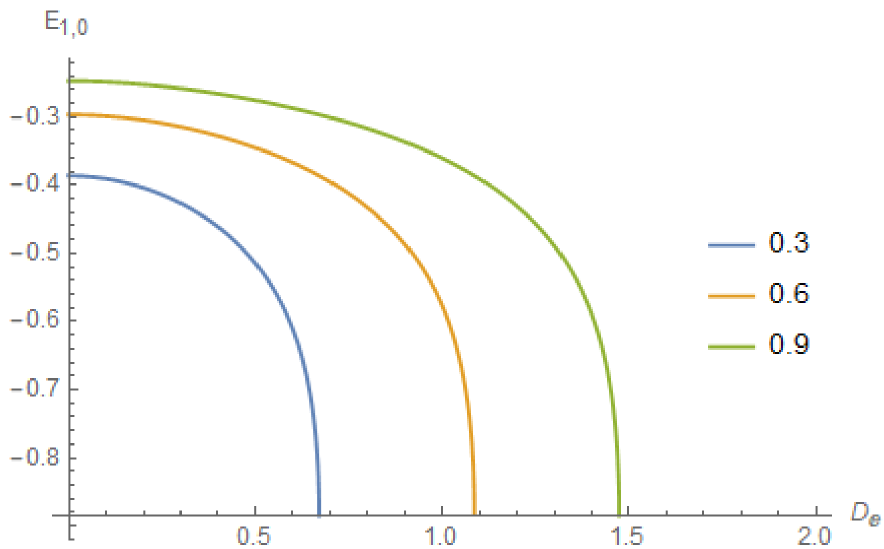

Figure 1.21: $E_{1,0}$ as a function of D_θ for $D_r = 0.3, 0.6$ and 0.9

only on the quantum number m, that make the corresponding bound state no longer exists. If we put $D_r = 0$, all the s-states($m = 0$) are absent because the critical value for D here is zero. We say here that the presence of radial dipole is essential for s-states to exist, otherwise the angular moment make them disappear. The same observation is made concerning the other m-states ($m > 0$), but the critical value of the angular moment is positive in all these cases and these critical values increase with m and also with the values of D_r ($Figure$ 1.25). This critical value is smaller for the sine states and this causes the spread of the spectrum of these states to be less than that of the cosine states on the axis of the angular momentum (figures 1.14 and 1.15 where the indice a is for *cosine* solutions and the b for *sine* ones). So the radial dipole has two effects, it moves the energies to higher values while enlarging the region of possible values of angular moment ($Figure$ 1.26)..The $Table$1.1 shows the critical values of D_θ for different values of D_r and m

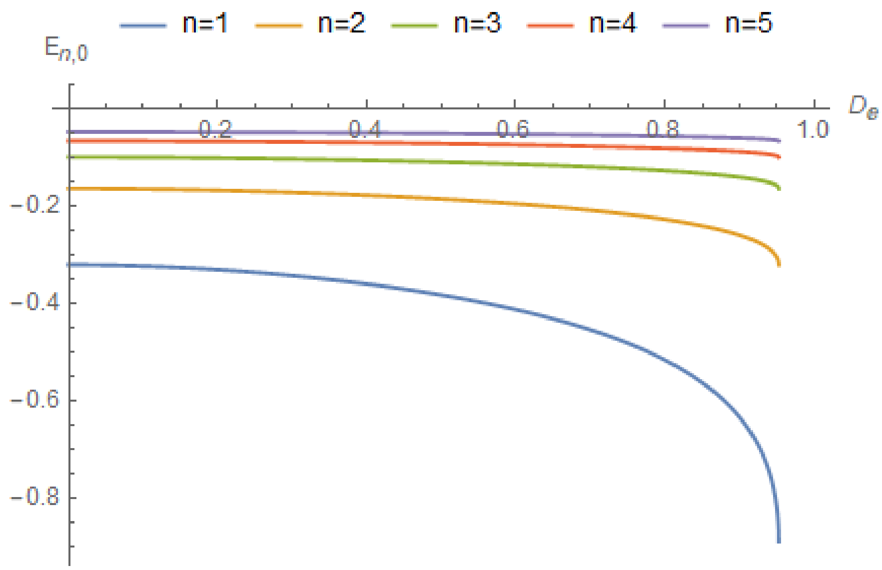

Figure 1.22: $E_{n,0}$ as a function of D_θ for $D_r = 0.5$ and $n = 1, 2, 3, 4$ and 5

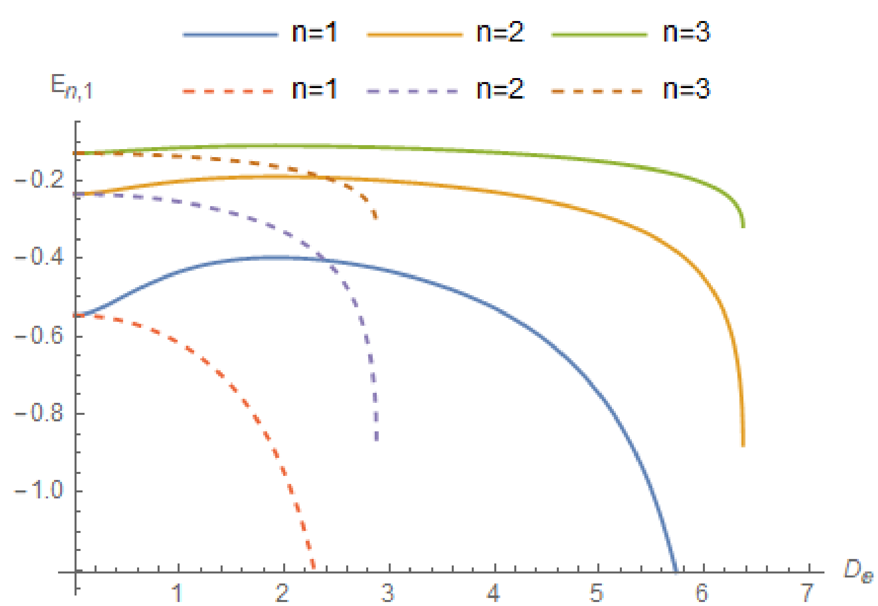

Figure 1.23: $E_{n,1}$ as a function of D_θ for $D_r = 0.5$ and $n = 1, 2, 3$ (se solutions are dashed)

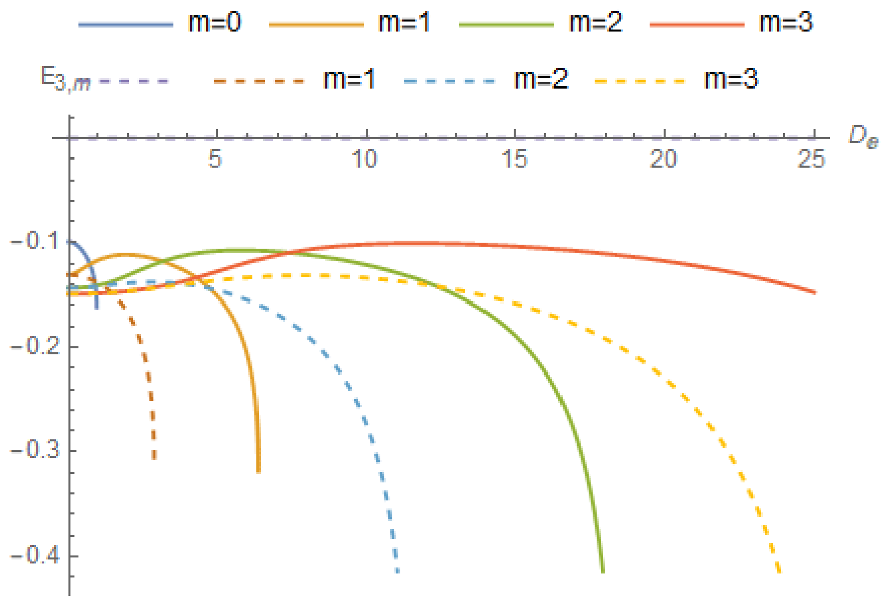

Figure 1.24: $E_{3,m}$ as a function of D_θ for$D_r = 0.5$ and $m = 0, 1, 2$ and 3 (se solutions are dashed)

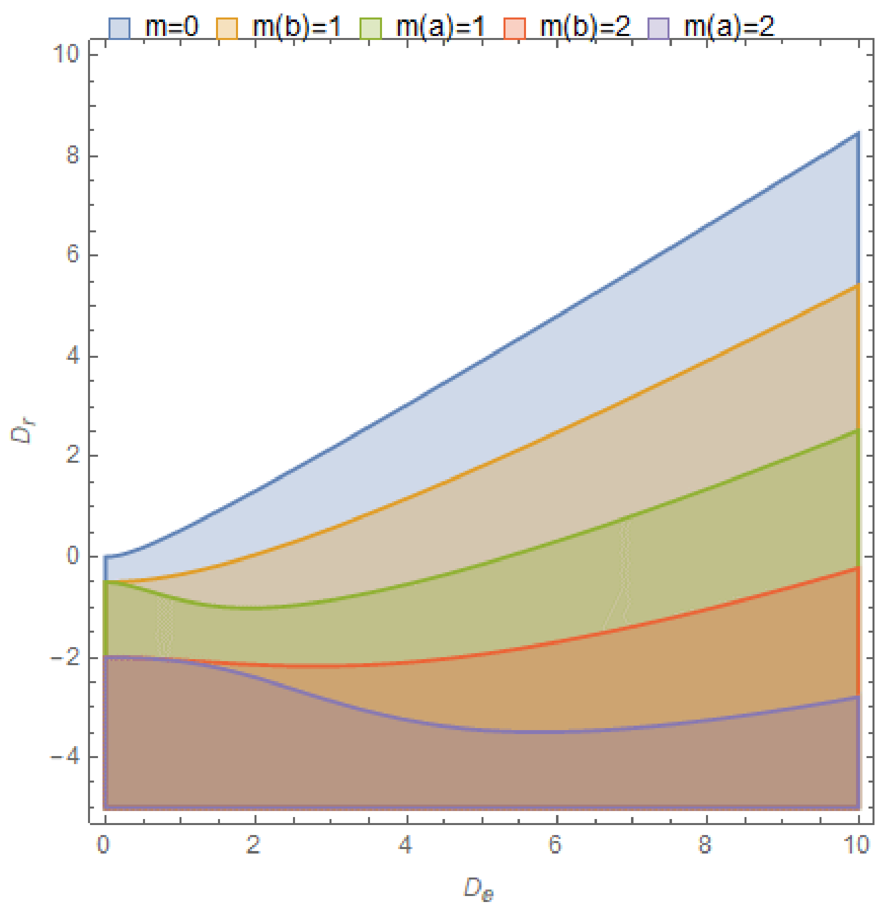

Figure 1.25: Forbidden regions of D_r and D_θ for$m = 0, 1, 2$ (a is for ce solutions and b for se ones)

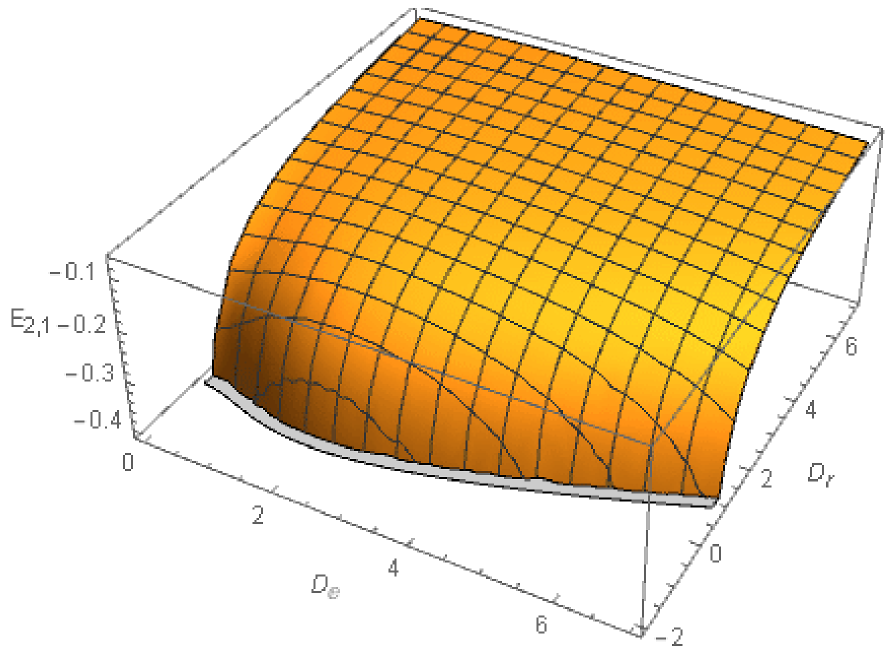

Figure 1.26: $E_{2,1}$ as a function of D_θ and D_r

From the graphes of effective potential we note that the dipole effecte to the bound state wile ,for the s-state the ce solution of dipole makes the state more bounded ($Figure$ 1.36),for the state $m = 1$ the se solution of dipole still makes the states more bounded then the ce solution which go rapidly to diffusion state as the angular momentum increas but the states of se solution become more bounded ($Figures$1.37, 1.38 and 1.39),for the state $m = 2$ for small angular momentum the two solution ce and se make the states Less bounding ,when the angular momentum increas the se state become more bounded but the ce state remaine less bounded ($Figures$ 1.40, 1.41 and 1.42) ,the state $m = 3$ regarding se and ce solutions of dipole is less bounded ($Figures$ 1.43, 1.44 and 1.45)

Case2 $V_3\left(r,\theta\right) = \mu\left[kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cos\theta\right)\right]$

This potential consists of pseudoharmonic potential PHO and dipole potential wille recently it was found to be one of those that best correspond to (quantum dots)QDs, for 2D disc-shaped quantum ring (QR) under the effect of an ionized donor atom, the conduction band electron is described by a PHO as a confinement potential and a donor impurity term where the angular equation with respect to it is the same of the case 1 and the constant of separation is the same E_θ equation 1.133 ,and the radial equation 1.17 in this case becomes

$$\frac{d^2R(r)}{dr^2} + \left[\left(E_\theta + \frac{1}{4} - \frac{2\mu^2 D_r}{\hbar^2}\right)\frac{1}{r^2} - \frac{2\mu^2 k}{\hbar^2}r^2 + \frac{2\mu E}{\hbar^2}\right]R(r) = 0 \tag{1.76}$$

From the radial equation we deduced the effective potential concern this case and plotted its variance in terms o r and m ($Figures$1.46, ..., 1.49)

And to solve this equation we followed the following steps where we put

$$r = a\sqrt{\rho} \tag{1.77}$$

So

$$\rho = \frac{r^2}{a^2} \Longrightarrow \frac{d\rho}{dr} = \frac{2r}{a^2} \Longrightarrow \frac{d\rho}{dr} = \frac{2\sqrt{\rho}}{a} \tag{1.78}$$

We calculate the derivative of $R(r)$ in terms of a new parameter ρ

The first derivative is

$$\frac{dR(r)}{dr} = \frac{2\sqrt{\rho}}{a}\frac{dR(r)}{d\rho} \tag{1.79}$$

The second derivative

$$\frac{d^2R(r)}{dr^2} = \frac{4\rho}{a^2}\frac{d^2R(\rho)}{d\rho^2} + \frac{2}{a^2}\frac{dR(\rho)}{d\rho} \tag{1.80}$$

To make the equation as a known equation we put the following changes

$$a^2 = \sqrt{\frac{\hbar^2}{2\mu^2 k}} \tag{1.81}$$

$$\varepsilon = \frac{2\mu E}{\hbar^2} \tag{1.82}$$

And

$$\eta = \left(E_\theta + \frac{1}{4} - \frac{2\mu^2 D_r}{\hbar^2}\right) \tag{1.83}$$

Then we substitute by the equation 1.79 to 1.83 in equation 1.76 so we get the following equation :

$$\frac{4\rho}{a^2}\frac{d^2R(\rho)}{d\rho^2} + \frac{2}{a^2}\frac{dR(\rho)}{d\rho} + \left[\frac{\eta}{\left(a\sqrt{\rho}\right)^2} - \frac{\left(a\sqrt{\rho}\right)^2}{a^4} + \varepsilon\right] R(\rho) = 0 \tag{1.84}$$

We divide the last equation by a^2 we find

$$4\rho\frac{d^2R(\rho)}{d\rho^2} + 2\frac{dR(\rho)}{d\rho} + \left(\frac{\eta}{\rho} - \rho + \varepsilon a^2\right) R(\rho) = 0 \tag{1.85}$$

To solve this equation, we use the following change:

$$R(\rho) = \rho^\alpha e^{-\rho/2}\omega\left(\rho\right) \tag{1.86}$$

Now we calculate the derivative of $R(\rho)$ in terms of a derivatives of a new function $\omega\left(\rho\right)$

The first derivative is

$$\frac{dR(\rho)}{d\rho} = \left(\alpha\rho^{\alpha-1}e^{-\rho/2} - \frac{1}{2}\rho^\alpha e^{-\rho/2}\right)\omega\left(\rho\right) + \rho^\alpha e^{-\rho/2}\frac{d\omega\left(\rho\right)}{d\rho} \tag{1.87}$$

The second derivative is

$$\frac{d^2R(\rho)}{d\rho^2} = \rho^{\alpha}e^{-\rho/2}\frac{d^2\omega(\rho)}{d\rho^2} + \left(2\alpha\rho^{\alpha-1}e^{-\rho/2} - \rho^{\alpha}e^{-\rho/2}\right)\frac{d\omega(\rho)}{d\rho} + \left(\alpha(\alpha-1)\rho^{\alpha-2}e^{-\rho/2} - \alpha\rho^{\alpha-1}e^{-\rho/2} + \frac{1}{4}\rho^{\alpha}e^{-\rho/2}\right)\omega(\rho) \tag{1.88}$$

From the equations 1.86 ,1.87 and 1.88 the equation 1.85 becomes

$$4\rho^{\alpha+1}e^{-\rho/2}\frac{d^2\omega(\rho)}{d\rho^2} + \left[2(4\alpha+1)\rho^{\alpha}e^{-\rho/2} - 4\rho^{\alpha+1}e^{-\rho/2}\right]\frac{d\omega(\rho)}{d\rho} + \left[4\alpha(\alpha-1)\rho^{\alpha-1}e^{-\rho/2} - 4\alpha\rho^{\alpha}e^{-\rho/2} + \rho^{\alpha+1}e^{-\rho/2} + 2\alpha\rho^{\alpha-1}e^{-\rho/2} - \rho^{\alpha}e^{-\rho/2} + \left(\tfrac{\eta}{\rho} - \rho + \varepsilon a^2\right)\rho^{\alpha}e^{-\rho/2}\right]\omega(\rho) = 0 \tag{1.89}$$

We dived the last equation by $4\rho^{\alpha}e^{-\rho/2}$,we find

$$\rho\frac{d^2\omega(\rho)}{d\rho^2} + \left[2\left(\alpha + \frac{1}{4}\right) - \rho\right]\frac{d\omega(\rho)}{d\rho} + \left[\left(\frac{1}{4}\alpha^2 + \frac{1}{4}\alpha\right)\rho^{-1} + \frac{1}{4}\rho + -\left(\frac{1}{4} + \alpha\right) - \frac{1}{4}\rho + \frac{\eta}{4\rho} + \frac{\varepsilon a^2}{4}\right]\omega(\rho) = 0 \tag{1.90}$$

So we get a new differential equation for $\omega(\rho)$:

$$\left[\rho\frac{d^2}{d\rho^2} + \left(2\left(\alpha + \frac{1}{4}\right) - \rho\right)\frac{d}{d\rho} + \frac{1}{\rho}\left(\left(\frac{1}{2}\alpha + \frac{1}{4}\right)^2 + \frac{4\eta - 1}{16}\right) + \frac{\varepsilon a^2}{4} - \left(\alpha + \frac{1}{4}\right)\right]\omega(\rho) = 0 \tag{1.91}$$

The last equation can be written as

$$\left[\rho\frac{d^2}{d\rho^2} + \left(2\left(\alpha + \frac{1}{4}\right) - \rho\right)\frac{d}{d\rho} + \frac{1}{\rho}\left(\left(\alpha - \frac{1}{4}\right)^2 + \frac{4\eta - 1}{16}\right) + \frac{\varepsilon a^2}{4} - \left(\alpha + \frac{1}{4}\right)\right]\omega(\rho) = 0 \tag{1.92}$$

Because α is a free parameter, we put:

$$\left(\alpha - \frac{1}{4}\right)^2 + \frac{4\eta - 1}{16} = 0 \tag{1.93}$$

Solving this latter equation for α yields two solutions:

$$\alpha = \frac{1}{4} \pm \frac{\sqrt{1 - 4\eta}}{4} \tag{1.94}$$

However since we require $\omega(\rho)$ to be a nonsingular function at $\rho = 0$, then the accepted value of α is:

$$\alpha = \frac{1}{4} + \frac{\sqrt{1-4\eta}}{4} \tag{1.95}$$

We put:

$$4n_r = \varepsilon a^2 - 4\alpha - 1;\ n_r = 0, 1, 2, ... \tag{1.96}$$

And this simplifies the equation 1.91 to be a hypergeometric equation :

$$\left(\rho\frac{d^2}{d\rho^2} + \left(2\alpha + \frac{1}{2} - \rho\right)\frac{d}{d\rho} + n_r\right)\omega(\rho) = 0 \tag{1.97}$$

And the solutions are the hypergeometric functions:

$$\omega(\rho) = N_1F_1(-n_r, 2\alpha + \frac{1}{2}, \rho)\ ;\ n_r = 0, 1, 2, ... \tag{1.98}$$

Here N is the normalized constant.

In terms of the variables r and θ, we can now write the general form of the radial wave function $R(r)$ by using 1.77 ,1.86and 1.98 as follows :

$$R(r) = N\left(\frac{r}{a}\right)^{2\alpha} e^{-\frac{r^2}{2a^2}}{}_1F_1\left(\left(\alpha + \frac{1}{4}\right) - \frac{\varepsilon a^2}{4}, 2\alpha + \frac{1}{2}, \frac{r^2}{a^2}\right) \tag{1.99}$$

For the energies we use the relations $\varepsilon = \frac{2\mu E}{\hbar^2}$,$\alpha = \frac{1}{4} + \frac{\sqrt{1-4\eta}}{4}$ and $\eta = \left(E_\theta + \frac{1}{4} - \frac{2\mu^2 D_r}{\hbar^2}\right)$ in $4n_r = \varepsilon a^2 - 4\alpha - 1$we find :

$$\frac{2\mu E}{\hbar^2}a^2 = 4n_r + \sqrt{1 - 4\left(E_\theta + \frac{1}{4} - \frac{2\mu^2 D_r}{\hbar^2}\right)} + 2 \tag{1.100}$$

We have $a^2 = \frac{\hbar}{\mu\sqrt{2k}}$ so the energy of the system is

$$E = \hbar\sqrt{2k}\left[2n_r + 1 + \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right] \tag{1.101}$$

$n_r = 0, 1, 2, ...$

When we substitute the parameters $\varepsilon, \alpha, \eta$ and $a^2$by its expressions in 1.99we find the radial wave function as:

$$R(r) = N_r\left(\frac{\mu\sqrt{2k}r^2}{\hbar}\right)^{\frac{1}{4}+\frac{1}{2}\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}} e^{-\frac{\mu\sqrt{2k}r^2}{2\hbar}}$$
$$ {}_1F_1\left(\frac{1}{2} + \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}} - \frac{E}{2\hbar\sqrt{2k}}, 1 + \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}, \frac{\mu\sqrt{2k}r^2}{\hbar}\right) \tag{1.102}$$

N_r is a constant of normalization

To find the final expression of energy of the system we substitute the constant of separation

1.34in energy expression 1.101 ,as

$$E_{3(n_r,m)} = \hbar\sqrt{2k}\left[2n_r + 1 + \sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{2\mu^2 D_r}{\hbar^2}}\right] \tag{1.103}$$

If we take the limit of the the harmonic oscillator when $D_r \longrightarrow 0 \text{and} \alpha \longrightarrow 0$ the energy becomes

$$E_{3(n_r,m)} = \hbar\sqrt{2k}\left[2n_r + 1 + |m|\right] \tag{1.104}$$

comparing with the energy of harmonic oscillator we find $2n_r + |m| = n \Longrightarrow n_r = \frac{1}{2}(n - |m|)$ where $n = 0, 1, 2, 3, ...$and $m = 0, 1, 2, 3, ...$,then the energy becomes as

$$E_{3(n_r,m)} = \hbar\sqrt{2k}\left[n - |m| + 1 + \sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{2\mu^2 D_r}{\hbar^2}}\right] \tag{1.105}$$

$c_{2m}\left(2\alpha\right)$ is characteristic values of Mathieu function

$$\begin{aligned} a_{(2m)}\left(2\alpha\right) = b_{(2m)}\left(2\alpha\right) &= 4m^2 + \frac{1}{2\left(4m^2-1\right)}\left(2\alpha\right)^2 + \frac{20m^2+7}{32\left(4m^2-1\right)^3\left(4m^2-4\right)}\left(2\alpha\right)^4 \\ &+ \frac{36m^4 + 232m^2 + 29}{64\left(4m^2-1\right)^5\left(4m^2-4\right)\left(4m^2-9\right)}\left(2\alpha\right)^6 + O\left(\left(2\alpha\right)^8\right) \end{aligned} \tag{1.106}$$

And $m = 0, 1, 2, ...$

In the Hartree units system and where $\left(\sqrt{2k} = \omega, \alpha = 2D_\theta\right)$ the last expression of energy becomes

$$E_{3(n_r,m)} = \left[n - |m| + 1 + \sqrt{\frac{1}{4}c_{2m}\left(4D_\theta\right) + 2D_r}\right] \tag{1.107}$$

The wave function of our system $\psi\left(r,\theta\right) = r^{-\frac{1}{2}}R(r)\Theta(\theta)$ is deduced from the angular part 1.35 and the radial part 1.102

$$\psi_3 = N\frac{\left(r\right)^{2\alpha-\frac{1}{2}}}{\left(a\right)^{2\alpha}}e^{-\frac{r^2}{2a^2}}\Theta(\theta)_1F_1\left(\left(\alpha+\frac{1}{4}\right) - \frac{\varepsilon a^2}{4}, 2\alpha + \frac{1}{2}, \frac{r^2}{a^2}\right) \tag{1.108}$$

When $a^2 = \sqrt{\frac{\hbar^2}{2\mu k}}$,$\varepsilon = \frac{2\mu E}{\hbar^2}$,$\alpha = \frac{1}{2}\left(\frac{1}{2} + \sqrt{1-4\eta}\right)$ and
$\eta = \left(-\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{1}{4} - \frac{2\mu^2 D_r}{\hbar^2}\right)$

For the potential $\mathbf{V}_4\left(r,\theta\right) = \frac{\mu}{q}\left[kr^2 + \left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_{19}\left(r,\theta\right)$ when we put $D_r \longrightarrow 0$ so

$$E_{4(n_r,m)} = \hbar\sqrt{2k}\left[n - |m| + 1 + \sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)}\right] \tag{1.109}$$

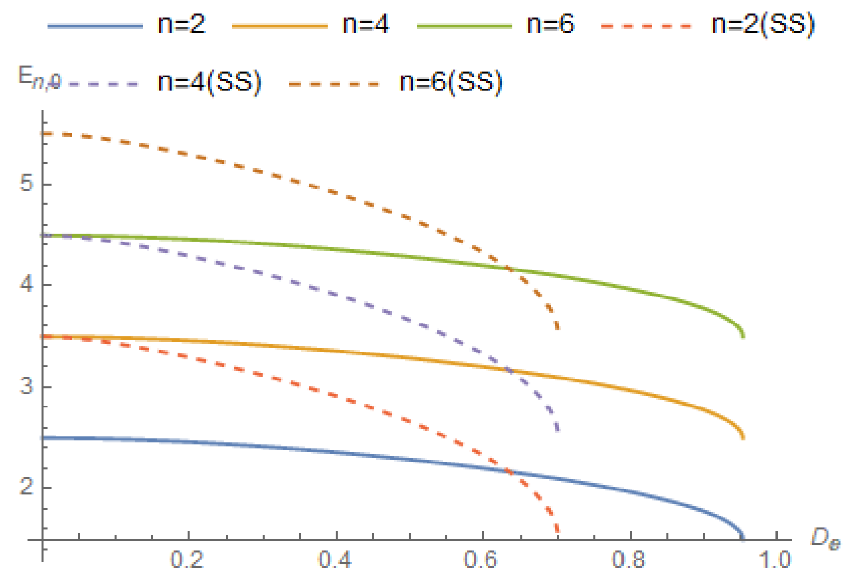

Figure 1.27: the non-relativistic energy and the non-relativistic limit of spin symmetry for $(PHO + dipole)$ potential $E_{n,0}$ (s $states$)in terms of D_θ

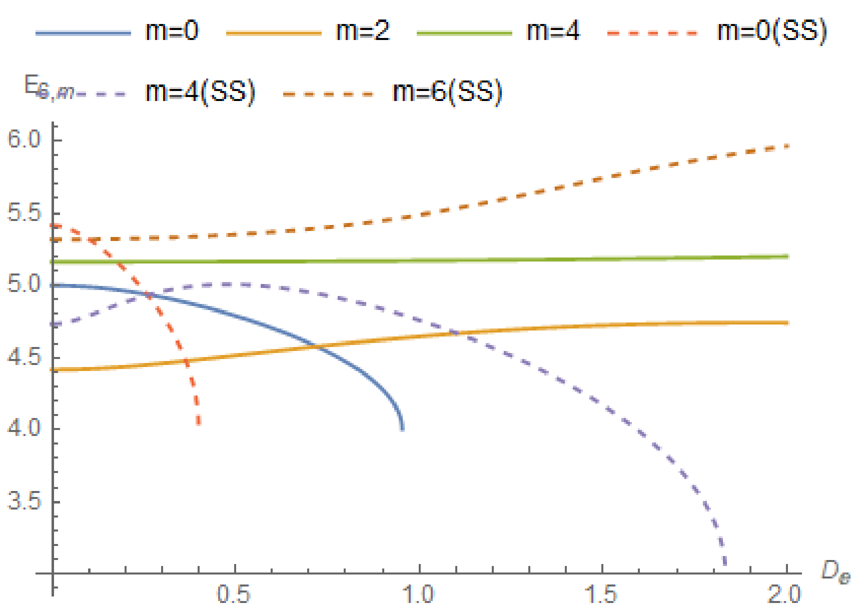

Figure 1.28: the non-relativistic energy and the non-relativistic limit of spin symmetry for $(PHO + dipole)$ potential $E_{6,m}$ in terms of D_θ

$n_r = 0, 1, 2, ...,$and $m = 0, 1, 2, ...$

The angular wave function is

$$\psi_4 = N(\frac{(r)^{2\alpha-\frac{1}{2}}}{(a)^{2\alpha}} e^{-\frac{r^2}{2a^2}} \Theta(\theta)_1 F_1 \left(\left(\alpha + \frac{1}{4} \right) - \frac{\varepsilon a^2}{4}, 2\alpha + \frac{1}{2}, \frac{r^2}{a^2} \right) \tag{1.110}$$

When $a^2 = \sqrt{\frac{\hbar^2}{2\mu k}}$,$\varepsilon = \frac{2\mu E}{\hbar^2}$,$\alpha = \frac{1}{2}\left(\frac{1}{2} + \sqrt{1-4\eta}\right)$ and
$\eta = \left(-\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{1}{4}\right)$

We atemted to apply these results to the two-dimensional QR we use the notations of [[30],[31], [32]]. where the energy is written as

$$E_{QR(n_r,m)} = \hbar\omega_0 \left[n - |m| + 1 + \sqrt{\frac{1}{4} c_{2(m)} \left(\frac{4\mu}{\varepsilon_r \hbar^2} D_\theta \right) + \lambda^2} \right] \tag{1.111}$$

Focusing on the effects of the dipole moment on the energies, we see in their expression 1.111 that the main modification is due to the parameter c_{2m} which replaces m. So we will study the effects on these energies through the root term:

$$\lambda_{ce\ ,se} = \sqrt{\frac{1}{4} c_{2(m)} \left(\frac{4\mu}{\varepsilon_r \hbar^2} D_\theta \right) + \lambda^2} \tag{1.112}$$

And especially through the correction that D_θ adds to this value and therefore we will focus on the dimensionless difference $\lambda_{ce\ ,se} - \lambda_0$ where $\lambda_0 = \sqrt{m^2 + \lambda^2}$ since will give us also the corrections of the energies in $\hbar\omega_0$ units. The indices ce and se in 1.112 indicate that the corrections depend on the chosen solution type for the angular equation 1.111.Parameter values used in our computations correspond to GaAs devices where $\lambda = 2$, $\mu = 0.067 m_e$, $\varepsilon_r = 12.65$ [[30],[31], [32]] and we use the Hartree atomic units . For the energy numerical values, we have $\hbar\omega_0 \approx 0.1 \sim 1 eV$ and this means that the energies of the levels considered in our work ($n = 1$ *and* 2 and $m = 0$ *and* 1) are in the intervals 0.5 to 0.8 eV or 5 to 8 eV depending on the value of $\hbar\omega_0$. For D, we choose them in the range 1 to 10 $a.u.$ because it corresponds to the experimental values of most molecular systems .Because of the behavior of the Mathieu characteristic values a_{2m} and b_{2m}, the corrections for the ce states $m = 0$ and the se states $m = 1$ are negative, while they are positive for all the other states for bothce and se solutions ($m = 0$ states exist only for ce solutions). Their values decrease with increasing m and those corresponding to ce solutions are larger than those of se ones for the same quantum numbers (*Figures* 1.29 *and* 1.30). These figures show that we can neglect the modifications for $m \geq 2$ as they are 10^2 smaller than those corresponding to the s-states ($m = 0$) and so they give corrections of the order of $10^{-5} eV$ or less. Depending on the values o f$\hbar\omega_0$ mentioned above, the energy corrections for $m = 0$ are around $10^{-3} eV$ while those corresponding to $m = 1$ are just a little bit smaller for ce states and approximatively equal to $10^{-4} eV$ for se states.Since these corrections are not the same for the different values of m, the dipolar term modifies the transition energies between the levels; in(*Figures*1.31 *and* 1.32),

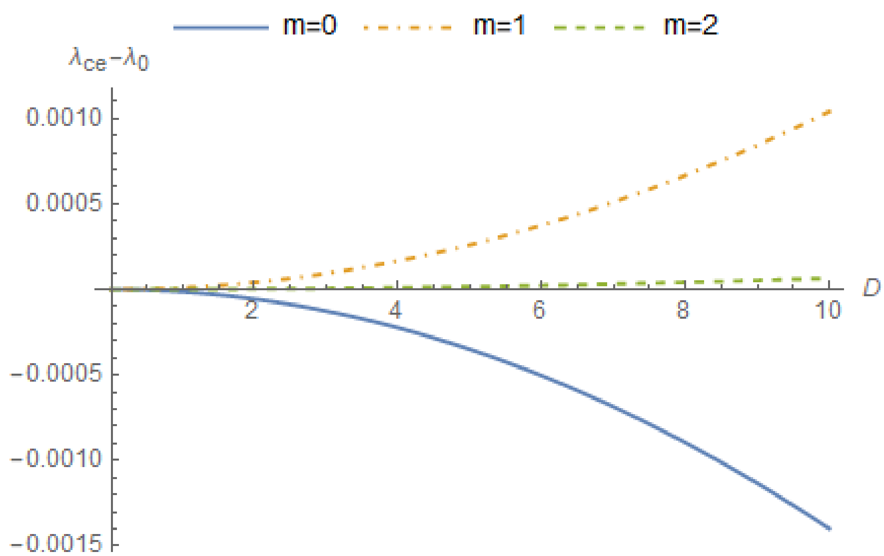

Figure 1.29: Corrections of ce energies in $\hbar\omega_0$ units for$m = 0, 1, 2$

we give as an example the effects on the transitions $(n, 1) \to (n, 0)$ and $(n, 2) \to (n, 1)$. Note that the presence of the dipole term increases the transition energies by more than 1% in the case of $(n, 1) \to (n, 0)$ while it decreases that of $(n, 2) \to (n, 1)$ by about 0.1%; this concerns ce states. Regarding the se states, its presence increases the energy of the $(n, 2) \to (n, 1)$ transition by less than 0.04

From 1.112, we see also that the corrections increase with the ratio μ/ε_r and thus they are more pronounced for the compounds $Ga_{1-x}Al_xAs$,since the effective mass for these materials is given by the formula $\mu = (0.067 + 0.085x)m_e$ with x real [47]. We show in $(Figures$ $1.33, 1.34$ and $1.35)$, these changes for $x = 0.3$ used in [[43], [47]] and also for the parameters of $CdSe$ $\mu/\varepsilon_r = 0.13/9.3$ studied in [42].We observe that the dipole corrections are 2 times greater for the $Ga_{1-x}Al_xAs$ than for the $GaAs$ and they are 7 times more pronounced than the latter in the case of $CdSe$

For the effective potential of this case pseudoharmonic plus dipole $(Figures 1.46\ to\ 1.49)$ we not that all states are confemed by the oscillator and the bounded stats not affected by the dipole potential or kratzer potential whoever the energy level or the momentum of this stats

Case 3: $V_5(r, \theta) = \mu\left[-\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha \sin^2\theta + \beta \sin\theta + \gamma\right)\cos^{-2}\theta\right]$

this potential at limit becomes what is called ring-shaped potential and its variations in terms of r and θ are shown in the graphs of $(Figures 1.5\ and\ 1.6)$

For this case the angular equation 1.16becomes

$$\frac{d^2\Theta}{d\theta^2} - \left(\alpha \sin^2\theta + \beta \sin\theta + \gamma\right)\cos^{-2}\theta\Theta - E_\theta\Theta = 0 \tag{1.113}$$

We make the following substitutions: $y = \frac{1-\sin\theta}{2}$ and $\Theta = y^\rho(1-y)^\sigma T$ in the last equation

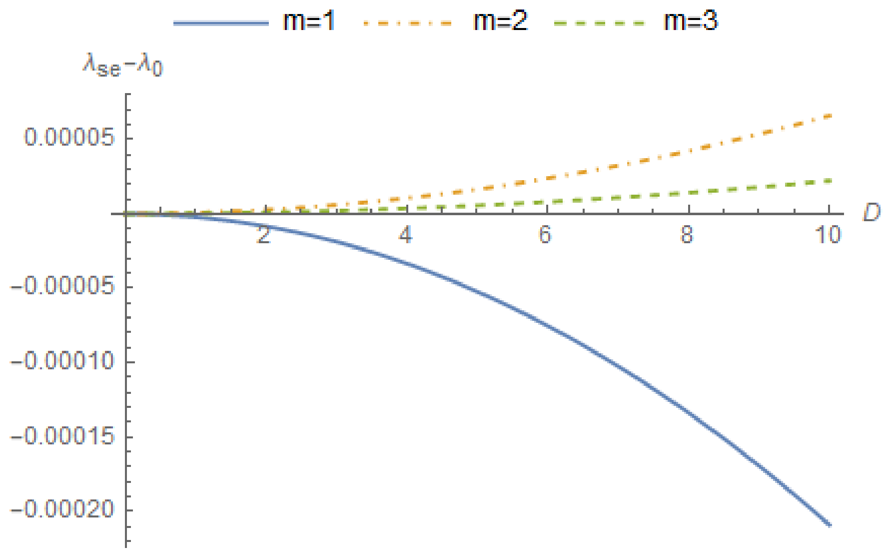

Figure 1.30: Corrections of se energies in $\hbar\omega_0$ units for $m = 1, 2, 3$

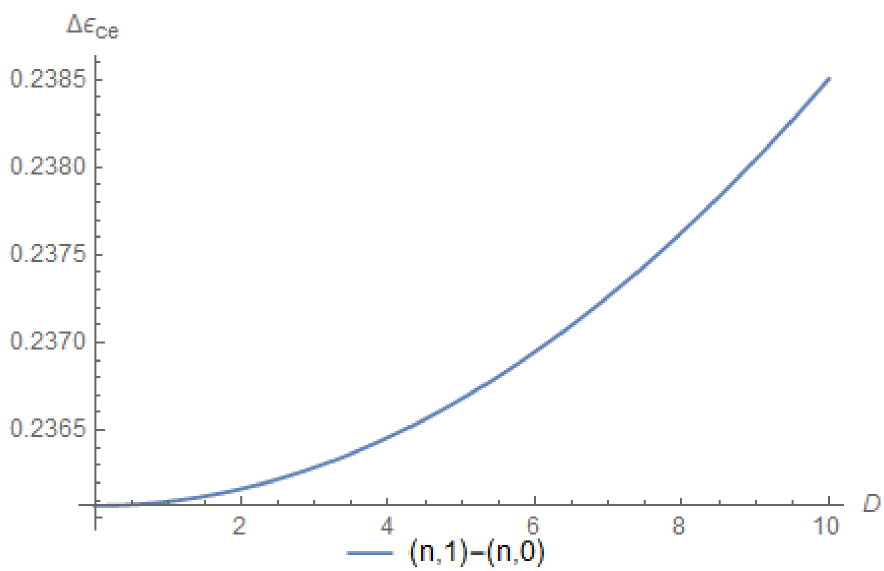

Figure 1.31: Correction of the transitions $(n, 1) \rightarrow (n, 0)$in $\hbar\omega_0$ units for ce solutions

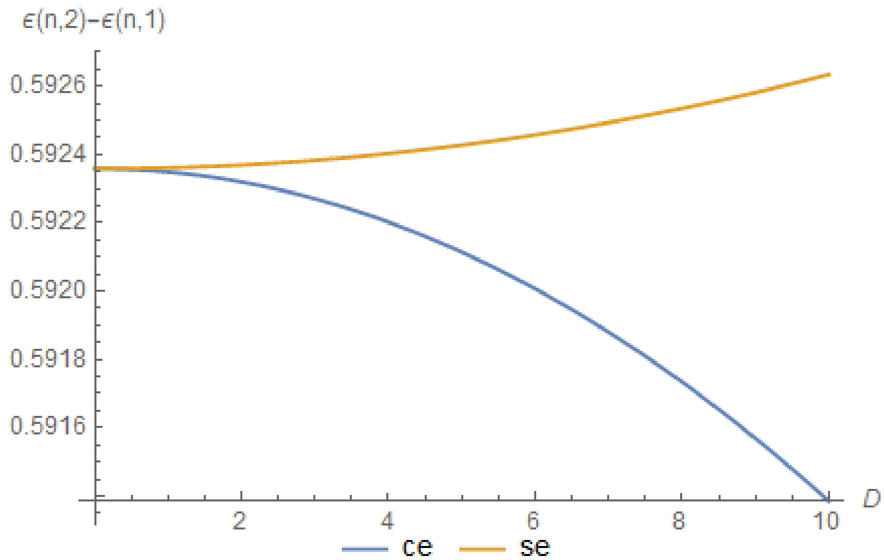

Figure 1.32: Correction of the transitions $(n, 2) \to (n, 1)$in $\hbar\omega_0$ units for ce and se solutions

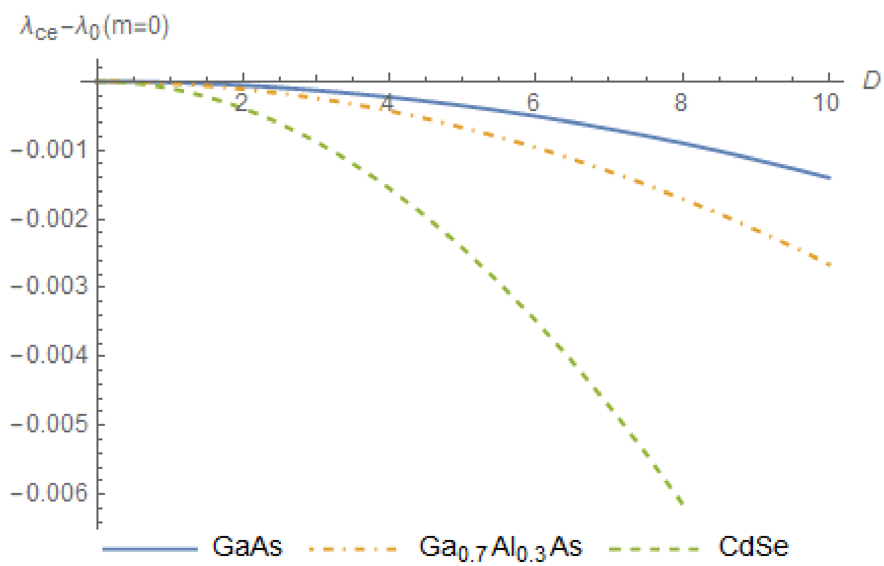

Figure 1.33: Corrections for some materials of ce energies in $\hbar\omega_0$ units for $m = 0$

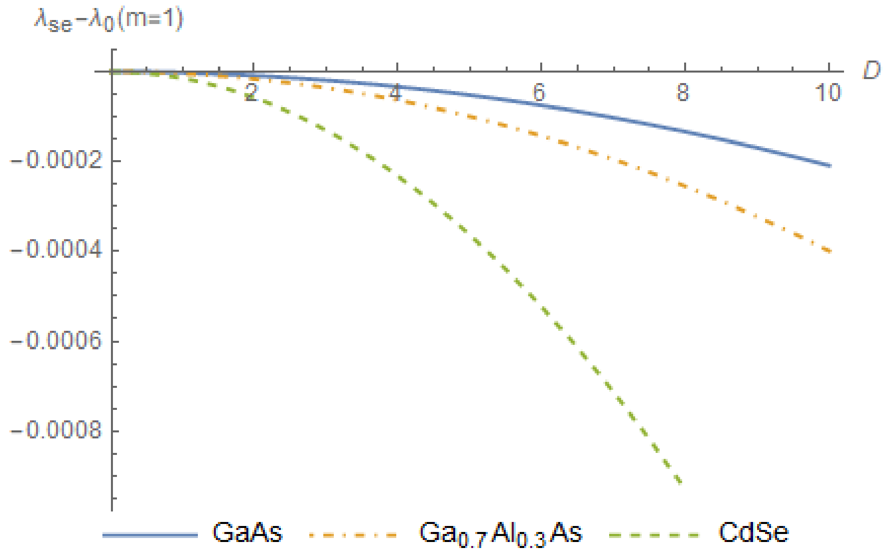

Figure 1.34: Corrections for some materials of ce energies in $\hbar\omega_0$ units for $m = 1$

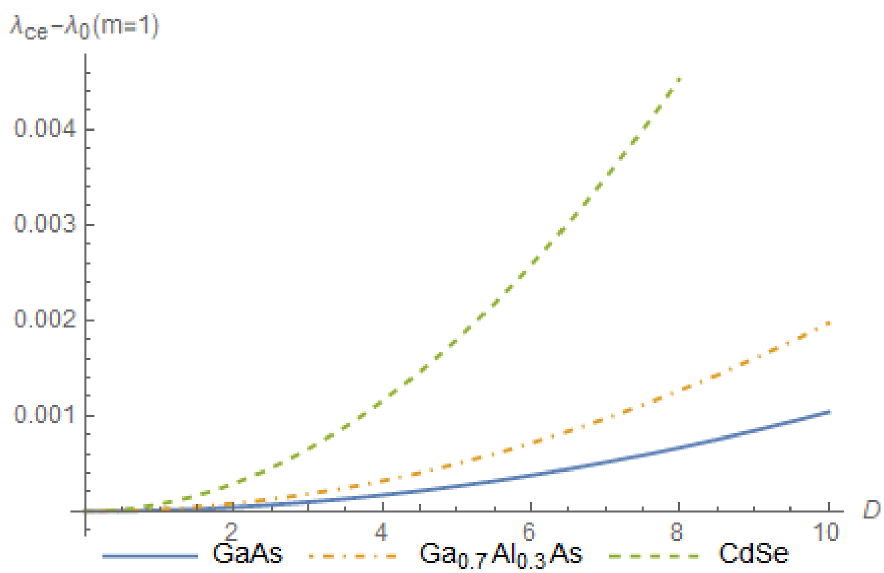

Figure 1.35: Corrections for some materials of ce energies in $\hbar\omega_0$ units for $m = 1$

,so we have to compute all parts of the equation by the new variable

$$y=\frac{1-\sin\theta}{2}\Longrightarrow\sin\theta=1-2y\Longrightarrow\sin^2\theta=(1-2y)^2 \tag{1.114}$$

And

$$\cos^2\theta=4y\,(1-y) \tag{1.115}$$

The first derivative of Θ with respect to θ is

$$\frac{d\Theta}{d\theta}=-\frac{\cos\theta}{2}\frac{d\Theta}{dy} \tag{1.116}$$

The second derivative of Θ with respect to θ is

$$\frac{d^2\Theta}{d\theta^2}=\frac{1-2y}{2}\frac{d\Theta}{dy}+y\,(1-y)\,\frac{d^2\Theta}{dy^2} \tag{1.117}$$

The first derivative of Θ with respect to a new variable y

$$\frac{d\Theta}{dy}=\left(\rho y^{\rho-1}(1-y)^{\sigma}-\sigma y^{\rho}(1-y)^{\sigma-1}\right)T+y^{\rho}(1-y)^{\sigma}\frac{dT}{dy} \tag{1.118}$$

The second derivative of Θ with respect to a new variable y

$$\begin{aligned}\frac{d^2\Theta}{dy^2} &= \left[\left(\rho\,(\rho-1)\,y^{\rho-2}(1-y)^{\sigma}-2\rho\sigma y^{\rho-1}(1-y)^{\sigma-1}+\sigma\,(\sigma-1)\,y^{\rho}(1-y)^{\sigma-1}\right)+\right]T\\ &+2\left(\rho y^{\rho-1}(1-y)^{\sigma}-\sigma y^{\rho}(1-y)^{\sigma-1}\right)\frac{dT}{dy}+y^{\rho}(1-y)^{\sigma}\frac{d^2T}{dy^2}\end{aligned} \tag{1.119}$$

By substituting the results 1.114to 1.118 in equation 1.113we find a new angular equation

$$y\,(1-y)\,\frac{d^2\Theta}{dy^2}+\frac{1-2y}{2}\frac{d\Theta}{dy}-\left[\left(\alpha\left((1-2y)^2\right)+\beta\,(1-2y)+\gamma\right)(4y\,(1-y))^{-2}-E_\theta\right]\Theta=0 \tag{1.120}$$

By using the equations 1.118 and 1.119the equation 1.120 becomes

$$\begin{gathered}y^{\rho+1}(1-y)^{\sigma+1}\frac{d^2T}{dy^2}+\left[2\left(\rho y^{\rho}(1-y)^{\sigma+1}-\sigma y^{\rho+1}(1-y)^{\sigma}\right)+\frac{1-2y}{2}y^{\rho}(1-y)^{\sigma}\right]\frac{dT}{dy}+\\ \left[\tfrac{1-2y}{2}\left(\rho y^{\rho-1}(1-y)^{\sigma}-\sigma y^{\rho}(1-y)^{\sigma-1}\right)+\right.\\ \left(\rho\,(\rho-1)\,y^{\rho-1}(1-y)^{\sigma+1}-2\sigma\rho y^{\rho}(1-y)^{\sigma}+\sigma\,(\sigma-1)\,y^{\rho+1}(1-y)^{\sigma}\right)\\ \left.+\left(-\left(\alpha\,(1-2y)^2+\beta\,(1-2y)+\gamma\right)(4y\,(1-y))^{-2}-E_\theta\right)y^{\rho}(1-y)^{\sigma}\right]T=0\end{gathered} \tag{1.121}$$

We divide the equation 1.121 on $y^{\rho}(1-y)^{\sigma}$ we find

$$\begin{gathered} y(1-y)\frac{d^2T}{dy^2}+2\left(\rho(1-y)-\sigma y+\frac{1}{4}-\frac{y}{2}\right)\frac{dT}{dy}+ \\ \left[\frac{1-2y}{2}\left(\rho y^{-1}-\sigma(1-y)^{-1}\right)+\left(\rho\left(\rho-1\right)y^{-1}(1-y)-2\sigma\rho+\sigma\left(\sigma-1\right)y\right)\right. \\ \left.+\left(-\left(\alpha\left(1-2y\right)^2+\beta\left(1-2y\right)+\gamma\right)\left[4y\left(1-y\right)\right]^{-2}-E_{\theta}\right)\right]T=0 \end{gathered} \tag{1.122}$$

When we take

$$\rho=\frac{1}{4}+\frac{1}{4}(1+4\alpha+4\beta+4\gamma)^{1/2} \tag{1.123}$$

$$\sigma=\frac{1}{4}+\frac{1}{4}(1+4\alpha-4\beta+4\gamma)^{1/2} \tag{1.124}$$

Thus the equation 1.122 becomes

$$y(1-y)\frac{d^2T}{dy^2}+\left[\left(2\rho+\frac{1}{2}\right)-\left(2\rho+2\sigma+1\right)y\right]\frac{dT}{dy}-\frac{1}{2}\left(-2E+\rho+\sigma+\rho\sigma+\gamma-\alpha\right)T=0 \tag{1.125}$$

The last equation is a hypergeometric equation type and its solution is hypergeometric function [3][36] :

$$T=F\left(2\rho,2\sigma,(2\rho+\frac{1}{2});y\right) \tag{1.126}$$

From the asymptotic behavior of the confluent series ($r\to\infty\Longrightarrow F=0$) which lead to $T\to 0$ when $r\to\infty$ we find the general condition of quantization :

$$2\rho=-m, m=0,1,2,... \tag{1.127}$$

We use 1.123,so

$$2\rho+m=0\Longrightarrow m+\frac{1}{2}+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}=0 \tag{1.128}$$

From 1.124 and 1.128 we find

$$2\sigma=m+1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}+\frac{1}{2}(1+4\alpha-4\beta+4\gamma)^{1/2} \tag{1.129}$$

And

$$2\rho+\frac{1}{2}=\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}+1 \tag{1.130}$$

So we can write the hypergeometric equation as

$$T = F(-m, m+1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}+\frac{1}{2}(1+4\alpha-4\beta+4\gamma)^{1/2}; 1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}; y) \tag{1.131}$$

From the form of the hypergeometric equation

$$4\rho\sigma = -\frac{1}{2}\left[2E_\theta + \rho + \sigma + 4\rho\sigma + \gamma - \alpha\right] \Longrightarrow 8\rho\sigma = -2E_\theta - \rho - \sigma - 4\rho\sigma - \gamma + \alpha \tag{1.132}$$

This require that

$$E_\theta = \alpha - \left[m + \frac{1}{2} + \frac{1}{4}(1 + 4\alpha + 4\beta + 4\gamma)^{1/2} + \frac{1}{4}(1 + 4\alpha - 4\beta + 4\gamma)^{1/2}\right]^2 \tag{1.133}$$

$m = 1, 2, 3, ...$

Which is the angular energy

We find the angular wave function when we substitute the function T in equation $\Theta = y^\rho(1 - y)^\sigma T$ as

$$\Theta(y) = y^\rho(1 - y)^\sigma F(-m, m + 1 + \frac{1}{2}(1 + 4\alpha + 4\beta + 4\gamma)^{1/2} + \frac{1}{2}(1 + 4\alpha - 4\beta + 4\gamma)^{1/2};$$
$$1 + \frac{1}{2}(1 + 4\alpha + 4\beta + 4\gamma)^{1/2}; y) \tag{1.134}$$

We use $y = \frac{1-\sin\theta}{2}$,so

$$\Theta(\theta) = \left(\frac{1 - \sin\theta}{2}\right)^\rho \left(\frac{1 + \sin\theta}{2}\right)^\sigma$$
$$F\left(\begin{array}{c} -m, m + 1 + \frac{1}{2}(1 + 4\alpha + 4\beta + 4\gamma)^{1/2} + \frac{1}{2}(1 + 4\alpha - 4\beta + 4\gamma)^{1/2}; \\ 1 + \frac{1}{2}(1 + 4\alpha + 4\beta + 4\gamma)^{1/2}; \frac{1-\sin\theta}{2} \end{array}\right) \tag{1.135}$$

$m = 0, 1, 2, ...$

We substitute the constant of separation E_θ 1.133 in the expression of the energy 1.64 we find the final expression of the energy of the system as

$$E_{n_r} = -2\frac{\mu^3 H^2}{\hbar^2}\left[2n_r + 1 + 2\right.$$
$$\left.\sqrt{-\alpha + \left[m + \frac{1}{2} + \frac{1}{4}(1 + 4\alpha + 4\beta + 4\gamma)^{1/2} + \frac{1}{4}(1 + 4\alpha - 4\beta + 4\gamma)^{1/2}\right]^2 + \frac{2\mu D_r}{\hbar^2}}\right]^{-2} \tag{1.136}$$

$n_r = 0, 1, 2, ...$,and $m = 0, 1, 2, ...$

We deduce the wave function of our system $\psi(r,\theta) = r^{-\frac{1}{2}}R(r)\Theta(\theta)$ from the angular part 1.135 and radial part 1.59

$$\psi_1 = Nr^{\frac{1}{2}+\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}} e_1^{-\sqrt{-\frac{2\mu E}{\hbar^2}}r} \left(\frac{1-\sin\theta}{2}\right)^\rho \left(\frac{1+\sin\theta}{2}\right)^\sigma$$
$$ {}_1F_1\left(\frac{1}{2}+\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}+\frac{\mu^2 H}{\hbar}\sqrt{-\frac{1}{2\mu E}}, 1+2\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}, 2\sqrt{-\frac{2\mu E}{\hbar^2}}r\right) \times$$
$$F\left(\begin{array}{c} -m, m+1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}+\frac{1}{2}(1+4\alpha-4\beta+4\gamma)^{1/2}; \\ 1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}; \frac{1-\sin\theta}{2} \end{array}\right) \tag{1.137}$$

When $\rho = \frac{1}{4}+\frac{1}{4}(1+4\alpha+4\beta+4\gamma)^{1/2}$ and $\sigma = \frac{1}{4}+\frac{1}{4}(1+4\alpha-4\beta+4\gamma)^{1/2}$

For the potential $\boldsymbol{V}_6(r,\theta) = \boldsymbol{\mu}\left[-\frac{H}{r}+\frac{1}{r}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\sin^2\theta+\beta\sin\theta+\gamma\right)\cos^{-2}\theta\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_1(r,\theta)$ when we put $D_r \longrightarrow 0$ so

The energy of system is

$$E_{n_r} = -2\frac{\mu^3 H^2}{\hbar^2}\left[2n_r+1+2\sqrt{-\alpha+\left[m+\frac{1}{2}+\frac{1}{4}(1+4\alpha+4\beta+4\gamma)^{1/2}+\frac{1}{4}(1+4\alpha-4\beta+4\gamma)^{1/2}\right]^2}\right]^{-2} \tag{1.138}$$

And the wave function is

$$\psi_1 = Nr^{\frac{1}{2}+\sqrt{-E_\theta}} e_1^{-\sqrt{-\frac{2\mu E}{\hbar^2}}r} \left(\frac{1-\sin\theta}{2}\right)^\rho \left(\frac{1+\sin\theta}{2}\right)^\sigma$$
$$ {}_1F_1\left(\frac{1}{2}+\sqrt{-E_\theta}+\frac{\mu^2 H}{\hbar}\sqrt{-\frac{1}{2\mu E}}, 1+2\sqrt{-E_\theta}, 2\sqrt{-\frac{2\mu E}{\hbar^2}}r\right) \times$$
$$F\left(\begin{array}{c} -m, m+1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}+\frac{1}{2}(1+4\alpha-4\beta+4\gamma)^{1/2}; \\ 1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}; \frac{1-\sin\theta}{2} \end{array}\right) \tag{1.139}$$

When $\rho = \frac{1}{4}+\frac{1}{4}(1+4\alpha+4\beta+4\gamma)^{1/2}$ and $\sigma = \frac{1}{4}+\frac{1}{4}(1+4\alpha-4\beta+4\gamma)^{1/2}$

Case 4: $V_7(r,\theta) = \mu\left[kr^2+\frac{D_r}{r^2}+\frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\sin^2\theta+\beta\sin\theta+\gamma\right)\cos^{-2}\theta\right]$

We substitute the constant of separation E_θ 1.133 in the energy expression 1.101 ,we find the final expression of the energy of the system as

$$E = \hbar\sqrt{2k}\,[2n_r + 1+$$
$$\sqrt{-\alpha + \left[m + \frac{1}{2} + \frac{1}{4}(1+4\alpha+4\beta+4\gamma)^{1/2} + \frac{1}{4}(1+4\alpha-4\beta+4\gamma)^{1/2}\right]^2 + \frac{2\mu^2 D_r}{\hbar^2}}\;\Bigg] \tag{1.140}$$

$n_r = 0, 1, 2, ...,$and$m = 0, 1, 2, ...$

We deduce the wave function of our system $\psi\,(r,\theta) = r^{-\frac{1}{2}}R(r)\Theta(\theta)$ from the angular part 1.135 and radial part 1.102

$$\psi_3 = N\left(\frac{1-\sin\theta}{2}\right)^{\rho}\left(\frac{1+\sin\theta}{2}\right)^{\sigma} r^{-\frac{1}{2}}\left(\frac{\mu\sqrt{2k}r^2}{\hbar}\right)^{\frac{1}{4}+\frac{1}{2}\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}} e^{-\frac{\mu\sqrt{2k}r^2}{2\hbar}}$$
$${}_1F_1\left(\frac{1}{2}+\frac{1}{2}\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}} - \frac{E}{2\hbar\sqrt{2k}}, 1+\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}, \frac{\mu\sqrt{2k}r^2}{\hbar}\right)\times$$
$$F\left(\begin{array}{c} -m, m+1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}+\frac{1}{2}(1+4\alpha-4\beta+4\gamma)^{1/2}; \\ 1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}; \frac{1-\sin\theta}{2} \end{array}\right) \tag{1.141}$$

When $\rho = \frac{1}{4}+\frac{1}{4}(1+4\alpha+4\beta+4\gamma)^{1/2}$and $\sigma = \frac{1}{4}+\frac{1}{4}(1+4\alpha-4\beta+4\gamma)^{1/2}$

For the potential $V_8\,(r,\theta) = \mu\left[kr^2 + \frac{1}{r}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\sin^2\theta + \beta\sin\theta + \gamma\right)\cos^{-2}\theta\right]$,we deduce the energy and wave function of this case from the energy and wave function of the last case above when we put $D_r \longrightarrow 0$ so

The energy expression is

$$E = \hbar\sqrt{2k}\,[2n_r + 1+$$
$$\sqrt{-\alpha + \left[m + \frac{1}{2} + \frac{1}{4}(1+4\alpha+4\beta+4\gamma)^{1/2} + \frac{1}{4}(1+4\alpha-4\beta+4\gamma)^{1/2}\right]^2}\;\Bigg] \tag{1.142}$$

$n_r = 0, 1, 2, ...,$and $m = 0, 1, 2, ...$

The angular wave function is

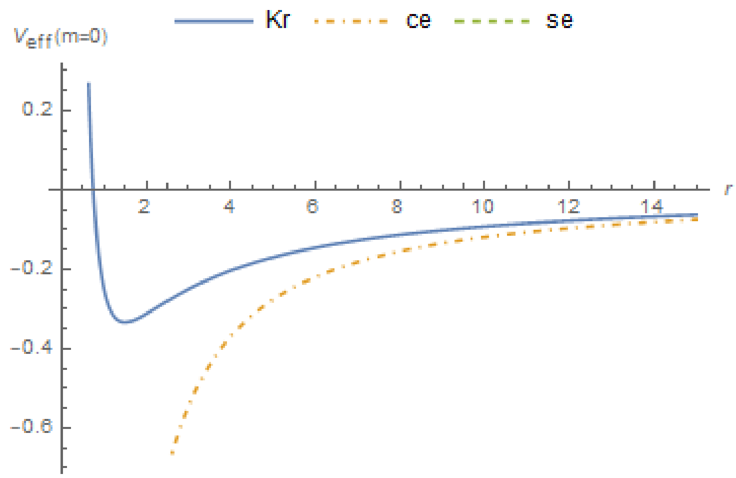

Figure 1.36: V_{eff} of(Kratzer+dipole) potential in terms of r for $m=0, D_r=1$ and $D_\theta=2$

$$\psi_8 = N\left(\frac{1-\sin\theta}{2}\right)^{\rho}\left(\frac{1+\sin\theta}{2}\right)^{\sigma} r^{-\frac{1}{2}}\left(\frac{\mu\sqrt{2k}r^2}{\hbar}\right)^{\frac{1}{4}+\frac{1}{2}\sqrt{-E_\theta}} e^{-\frac{\mu\sqrt{2k}r^2}{2\hbar}} \times$$
$${}_1F_1\left(\frac{1}{2}+\frac{1}{2}\sqrt{-E_\theta}-\frac{E}{2\hbar\sqrt{2k}}, 1+\sqrt{-E_\theta}, \frac{\mu\sqrt{2k}r^2}{\hbar}\right)\times$$
$$F\left(\begin{array}{c} -m, m+1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}+\frac{1}{2}(1+4\alpha-4\beta+4\gamma)^{1/2}; \\ 1+\frac{1}{2}(1+4\alpha+4\beta+4\gamma)^{1/2}; \frac{1-\sin\theta}{2} \end{array}\right) \tag{1.143}$$

When $\rho=\frac{1}{4}+\frac{1}{4}(1+4\alpha+4\beta+4\gamma)^{1/2}$ and $\sigma=\frac{1}{4}+\frac{1}{4}(1+4\alpha-4\beta+4\gamma)^{1/2}$

Regarding the $case5$ to $case10$ the solution of angular equation is obtained by the same methode of the $case3$ and $case4$ as a solution of hypergeometric equation and the angular part of wave function is found as hypergeometric function ,the energy expression and the radial part of wave function is a same of case 1 for kratzer potential and is a same of $case2$ for pseudoharmonic potential, the results are shown in the ($Tables$ 1.1, ..., 1.7) below and the detailed calculation is provided in *Appendix*1

1.3 Relativistic Studies of 2D Non-Central Potentials

1.3.1 Introduction

In quantum mechanics, it is well known that the Schrödinger equation plays important roles for describing the behaviors of a particle at the microscopic scale. However, when the relativistic effect becomes important, the Schrödinger equation should be replaced with relativistic wave equations, i.e., the Klein–Gordon equation for spin-0 particles and the Dirac equation for spin-1/2 particles, Recently, many researchers have been working on the exact solution of

$f(\theta)$	E_θ
$\left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta$	$-\frac{1}{4}c_{2m}(2\alpha)$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\sin^2\theta+\beta\sin\theta+\gamma\right)\cos^{-2}$	$\alpha-\left[m+\frac{1}{2}+\frac{1}{4}(1+4\alpha+4\beta+4\gamma)^{1/2}+\frac{1}{4}(1+4\alpha-4\beta+4\gamma)\right]^2$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\frac{\theta}{2}+\beta\tan\frac{\theta}{2}+\gamma\right)$	$\alpha-\gamma-\frac{\left[m+\frac{1}{2}+\frac{1}{2}(1+16\alpha)^{1/2}\right]-4\beta^2}{4\left[m+\frac{1}{2}+\frac{1}{2}(1+16\alpha)\right]^2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cot^2\frac{\theta}{2}+\beta\cot\frac{\theta}{2}+\gamma\right)$	$\alpha-\gamma-\frac{\left[m+\frac{1}{2}+\frac{1}{2}(1+16\alpha)^{1/2}\right]-4\beta^2}{4\left[m+\frac{1}{2}+\frac{1}{2}(1+16\alpha)\right]^2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\theta+\beta\tan\theta+\gamma\right)$	$\alpha-\gamma-\frac{\left[(1+4\alpha)^{1/2}+1+2m\right]^4-4\beta^2}{4\left[(1+4\alpha)^{1/2}+1+2m\right]^2}$

Table 1.2: The 2D constant of separation

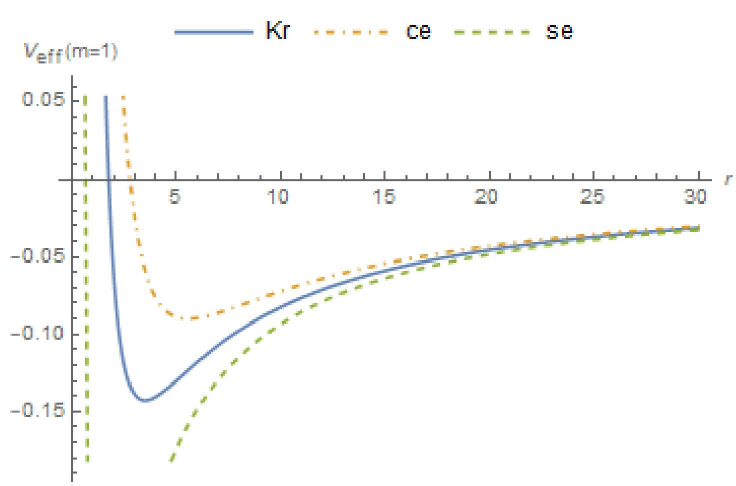

Figure 1.37: V_{eff} of(Kratzer+dipole) potential in terms of r for $m=1, D_r=1$ and $D_\theta=2$

$f(\theta)$	$\Theta(\theta)$
$\left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta$	$Mathieu function$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\sin^2\theta+\beta\sin\theta+\gamma\right)\cos^{-2}$	$\left(\dfrac{1-\sin\theta}{2}\right)^{\rho}\left(\dfrac{1+\sin\theta}{2}\right)^{\sigma}F\left(2\rho,2\sigma,(2\rho+\frac{1}{2});\frac{1-\sin\theta}{2}\right)$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\dfrac{\theta}{2}+\beta\tan\dfrac{\theta}{2}+\gamma\right)$	$-e^{i\rho\theta}(1+e^{i\theta})^{\sigma}F\left(2\rho,2\sigma,(2\rho+1);-e^{i\theta}\right)$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cot^2\dfrac{\theta}{2}+\beta\cot\dfrac{\theta}{2}+\gamma\right)$	$(-1)^{i\rho+1}e^{i\rho\theta}(1-e^{i\theta})^{\sigma}F\left(2\rho,2\sigma,(2\rho+1);e^{i\theta}\right)$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\theta+\beta\tan\theta+\gamma\right)$	$\left(1+e^{2i\theta\rho}\right)\left(-e^{2i\theta}\right)^{\sigma}F\left(2\rho,2\sigma,1+(1+4\alpha)^{1/2};1+e^{2i\theta}\right)$

Table 1.3: The 2D angular part of wave function

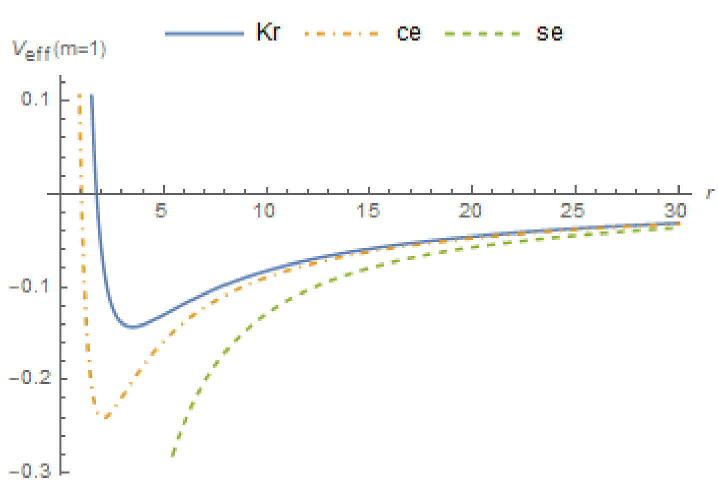

Figure 1.38: V_{eff} of(Kratzer+dipole) potential in terms of r for $m=1, D_r=1$ and $D_\theta=5$

$f(\theta)$	ρ	σ
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\sin^2\theta+\beta\sin\theta+\gamma\right)\cos^{-2}$	$\frac{1}{4}+\frac{1}{4}(1+4\alpha+4\beta+4\gamma)^{1/2}$	$\frac{1}{4}+\frac{1}{4}(1+4\alpha-4\beta+4\gamma)^{1/2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\frac{\theta}{2}+\beta\tan\frac{\theta}{2}+\gamma\right)$	$\rho=(-E_\theta+\alpha-i\beta-\gamma)^{1/2}$	$\sigma=\frac{1}{2}+\frac{1}{2}(1+16\alpha)^{1/2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cot^2\frac{\theta}{2}+\beta\cot\frac{\theta}{2}+\gamma\right)$	$\rho=(-E_\theta+\alpha-i\beta-\gamma)^{1/2}$	$\sigma=\frac{1}{2}+\frac{1}{2}(1+16\alpha)^{1/2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\tan^2\theta+\beta\tan\theta+\gamma)$	$\frac{1}{2}+\frac{1}{2}(1+4\alpha)^{1/2}$	$\frac{1}{2}(-E_\theta+\alpha-i\beta-\gamma)^{1/2}$

Table 1.4: The parameters of 2D constant of separation

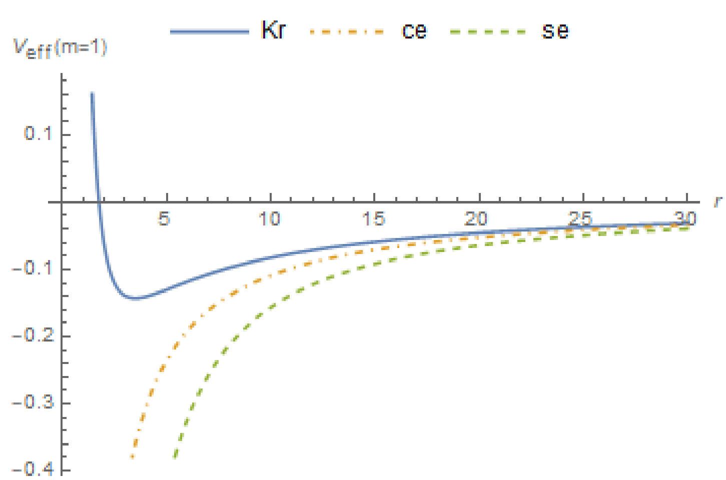

Figure 1.39: V_{eff} of(Kratzer+dipole) potential in terms of r for $m=1, D_r=1$ and $D_\theta=7$

$V(r)$	$R(r)$	λ	β^2
$-\frac{H}{r}+\frac{D_r}{r^2}$	$N_r r^\lambda e^{-\beta r}\ {}_1F_1\left(-n_r, 2\lambda, 2\beta r\right)$	$\frac{1}{2}+\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}$	$-\frac{2\mu E}{\hbar^2}$
$-\frac{H}{r}$	$N_r r^\lambda e^{-\beta r}\ {}_1F_1\left(-n_r, 2\lambda, 2\beta r\right)$	$\frac{1}{2}+\sqrt{-E_\theta}$	$-\frac{2\mu E}{\hbar^2}$
$kr^2+\frac{D_r}{r^2}$	$_r\left(\frac{r}{\beta}\right)^{\frac{1}{2}+\frac{\sqrt{1-4\lambda}}{2}} e^{-\frac{r^2}{2\beta^2}}\times$ ${}_1F_1\left(-n_r, 1+\frac{\sqrt{1-4\lambda}}{2}, \frac{r^2}{\beta^2}\right)$	$E_\theta+\frac{1}{4}-\frac{2\mu^2 D_r}{\hbar^2}$	$\frac{\hbar}{\mu\sqrt{2k}}$
kr^2	$N_r\left(\frac{r}{\beta}\right)^{\frac{1}{2}+\frac{\sqrt{1-4\lambda}}{2}} e^{-\frac{r^2}{2\beta^2}}\times$ ${}_1F_1\left(-n_r, 1+\frac{\sqrt{1-4\lambda}}{2}, \frac{r^2}{\beta^2}\right)$	$E_\theta+\frac{1}{4}$	$\frac{\hbar}{\mu\sqrt{2k}}$

Table 1.5: The radial part of 2D wave function

$V(r)$	E_{n_r}
$-\frac{H}{r}+\frac{D_r}{r^2}$	$-2\frac{\mu^3 H^2}{\hbar^2}\left(2n_r+2\sqrt{-E_\theta+\frac{2\mu D_r}{\hbar^2}}+1\right)^{-2}$
$-\frac{H}{r}$	$-2\frac{\mu^3 H^2}{\hbar^2}\left(2n_r+2\sqrt{-E_\theta}+1\right)^{-2}$
$kr^2+\frac{D_r}{r^2}$	$\hbar\sqrt{2k}\left[2n_r+1+\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}\right]$
kr^2	$\hbar\sqrt{2k}\left[2n_r+1+\sqrt{-E_\theta}\right]$

Table 1.6: The 2D energy expression

$V(r,\theta)$	$\psi(r,\theta)$
$\mu\left(-\frac{H}{r}+\frac{D_r}{r^2}+\frac{f(\theta)}{r^2}\right)$	$Nr^{\lambda-\frac{1}{2}}e^{-\beta r}\ {}_1F_1\left(-n_r,2\lambda,2\beta r\right)\Theta\left(\theta\right)$
$\mu\left(-\frac{H}{r}+\frac{f(\theta)}{r^2}\right)$	$Nr^{\lambda-\frac{1}{2}}e^{-\beta r}\ {}_1F_1\left(-n_r,2\lambda,2\beta r\right)\Theta\left(\theta\right)$
$\mu\left(kr^2+\frac{D_r}{r^2}+\frac{f(\theta)}{r^2}\right)$	$Nr^{-\frac{1}{2}}\left(\frac{r}{\beta}\right)^{\frac{1}{2}+\frac{\sqrt{1-4\lambda}}{2}}e^{-\frac{r^2}{2\beta^2}}\,{}_1F_1\left(-n_r,1+\frac{\sqrt{1-4\lambda}}{2},\frac{r^2}{\beta^2}\right)\Theta\left(\theta\right)$
$\mu\left(kr^2+\frac{f(\theta)}{r^2}\right)$	$Nr^{-\frac{1}{2}}\left(\frac{r}{\beta}\right)^{\frac{1}{2}+\frac{\sqrt{1-4\lambda}}{2}}e^{-\frac{r^2}{2\beta^2}}\,{}_1F_1\left(-n_r,1+\frac{\sqrt{1-4\lambda}}{2},\frac{r^2}{\beta^2}\right)\Theta\left(\theta\right)$

Table 1.7: The 2D wave function

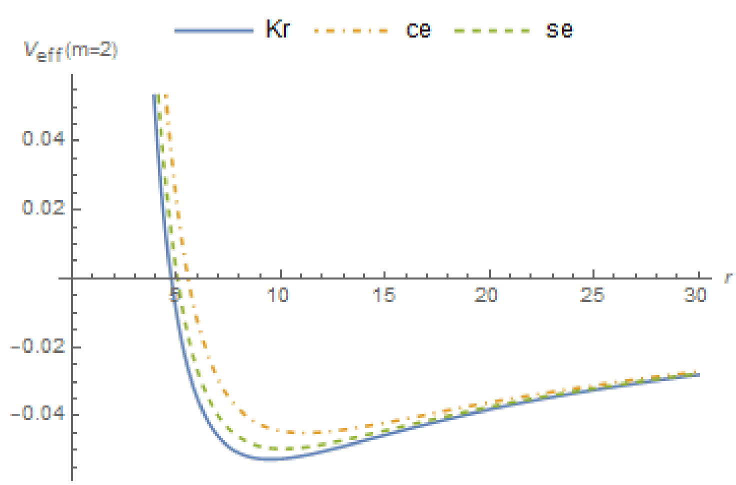

Figure 1.40: V_{eff} of(Kratzer+dipole) potential in terms of r for $m=2, D_r=1$ and $D_\theta=2$

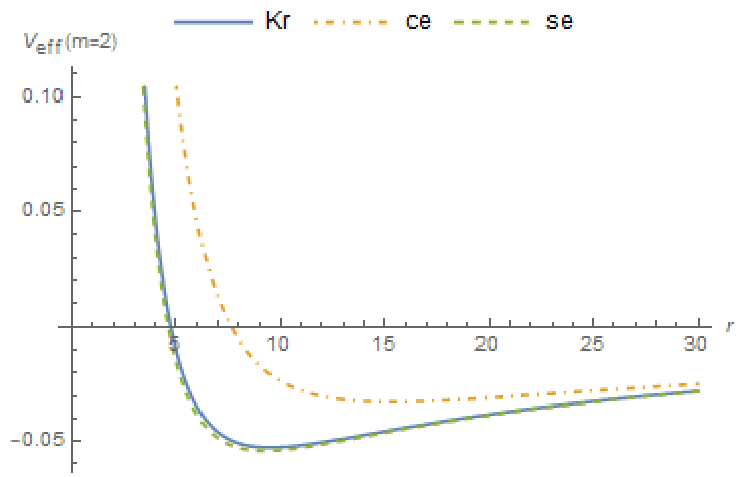

Figure 1.41: V_{eff} of(Kratzer+dipole) potential in terms of r for $m = 2, D_r = 1$ and $D_\theta = 5$

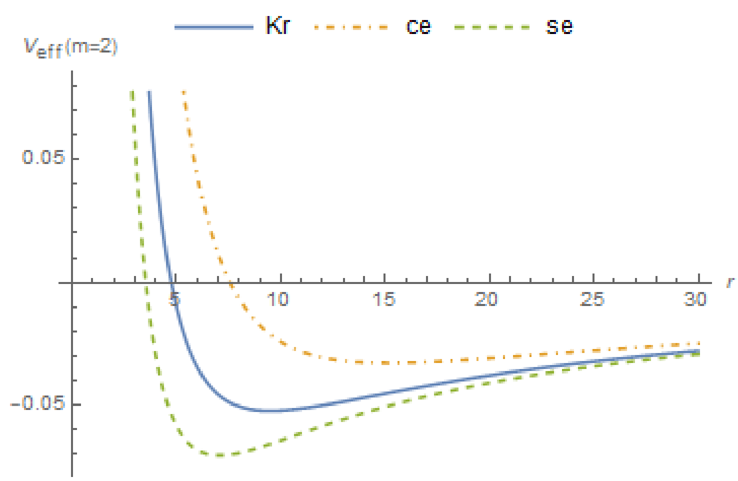

Figure 1.42: V_{eff} of(Kratzer+dipole) potential in terms of r for $m = 2, D_r = 1$ and $D_\theta = 7$

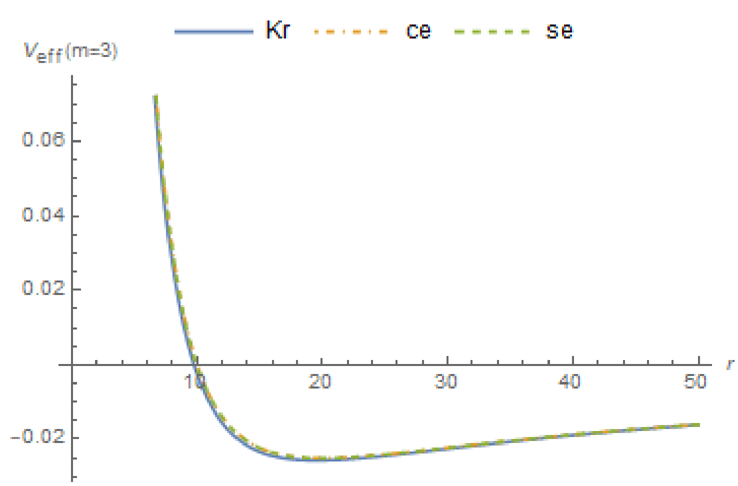

Figure 1.43: V_{eff} of(Kratzer+dipole) potential in terms of r for $m = 3, D_r = 1$ and $D_\theta = 2$

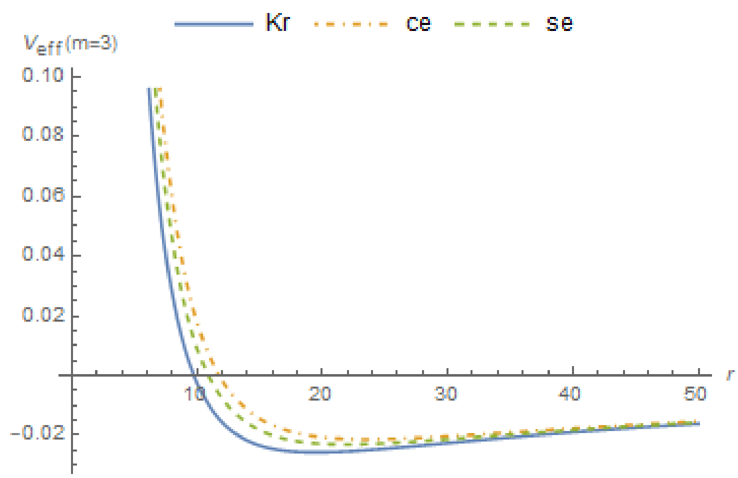

Figure 1.44: V_{eff} of(Kratzer+dipole) potential in terms of r for $m = 3, D_r = 1$ and $D_\theta = 5$

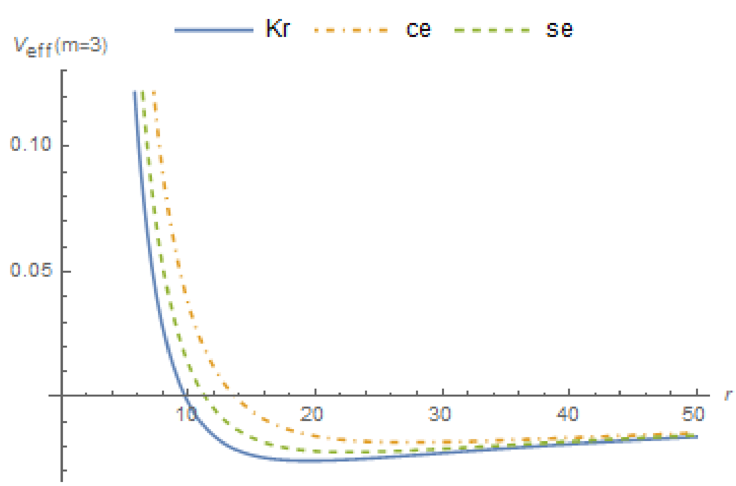

Figure 1.45: V_{eff} of(Kratzer+dipole) potential in terms of r for $m = 3, D_r = 1$ and $D_\theta = 7$

the Dirac equation with different non-central potentials [48] [49] [50] [51] [52],but there are no analytical solutions for the Klein-Gordon and Dirac equations for this potentials,then both equations reduce to the same Schrödinger type equation if we consider the cases of spin and pseudo-spin symmetries The near realization of these symmetries may explain degeneracies in some heavy meson spectra (spin symmetry) or in single-particle energy levels in nuclei (pseudospin symmetry) [53] [54] [55]. The spin and pseudospin symmetries are SU(2)-type symmetries of a Dirac Hamiltonian. They have been studied since 1969 in quasidegeneracy. Besides, these symmetries were considered in the context of deformed nuclei [56], the superdeformation [57] , the magnetic moment interpretation [58] [59] , the identical bands [60][61][62] [63] and the effective shell-model coupling scheme [64] . These symmetries were also used to study the relativistic theory of both central and ring-shaped Kratzer potentials [65][66].and the relativistic effects of a moving particle in the field of a pseudoharmonic oscillatory ring-shaped potential under the spin and pseudospin symmetric Dirac wave equation [67]

1.3.2 Klein-Gordon Equation

The stationary Klein-Gordon equation for a single charge q in both scalar $S(\overrightarrow{r})$and vector $U(\overrightarrow{r})$ potentials is written as:

$$\left[c^2p^2-(E-U(\overrightarrow{r}))^2+\left(\mu c^2-S(\overrightarrow{r})\right)^2\right]\psi(\overrightarrow{r})=0 \tag{1.144}$$

Spin or pseudo-spin symmetry are defined by the relation $S(\overrightarrow{r})=\pm U(\overrightarrow{r})$, we substitute it in equation 1.144,so

$$\left[c^2p^2-(E-U(\overrightarrow{r}))^2+\left(\mu c^2-\pm U(\overrightarrow{r})\right)^2\right]\psi(\overrightarrow{r})=0$$

The wave equation 1.144 reduce to the following second order equation:

$$\left[c^2p^2-2\left(E\pm\mu c^2\right)U(\overrightarrow{r})-\left(E^2-\mu^2c^4\right)\right]\psi(\overrightarrow{r})=0 \tag{1.145}$$

The equation is easily written as a Schrödinger equation with the transformations:

$$\left(\frac{E}{\mu c^2}\pm 1\right)U(\overrightarrow{r})\rightarrow U(\overrightarrow{r}) \text{ and } \frac{1}{2}\left(\frac{E^2}{\mu c^2}-\mu c^2\right)\rightarrow E \tag{1.146}$$

We dived the equation 1.145 by μc^2 ,we get

$$\left[\frac{c^2p^2}{\mu c^2}-2\left(\frac{E}{\mu c^2}\pm 1\right)U(\overrightarrow{r})-\left(\frac{E^2}{\mu c^2}-\mu c^2\right)\right]\psi(\overrightarrow{r})=0 \tag{1.147}$$

The equation is easily written as a Schrödinger equation with the transformations:

$$\left(\frac{E}{\mu c^2} \pm 1\right) U(\overrightarrow{r}) \rightarrow U(\overrightarrow{r}) \text{and } \frac{1}{2}\left(\frac{E^2}{\mu c^2} - \mu c^2\right) \rightarrow E \tag{1.148}$$

So we use the last transformation we find following equation

$$\left(\frac{p^2}{\mu} - 2U(\overrightarrow{r}) - 2E\right)\psi(\overrightarrow{r}) = 0 \tag{1.149}$$

When we divide the equation 1.149 by 2 ,we find the Schrödinger equation

$$\left[\frac{p^2}{2\mu} - (U(\overrightarrow{r}) + E)\right]\psi(\overrightarrow{r}) = 0 \tag{1.150}$$

Here we get a system where the potential depends on the energy. These energy dependent potentials have been considered for a long time when the relativistic effects began to be taken into account in quantum physics [68][103][70] and recently a lot of works are devoted to this type of potentials [71][72][73]

1.3.3 Dirac Equation

We consider now the stationary Dirac equation:

$$\left[c\overrightarrow{\alpha}\overrightarrow{p} + \beta\left(\mu c^2 + S(\overrightarrow{r})\right) - (E - U(\overrightarrow{r}))\right]\psi(\overrightarrow{r}) = 0 \tag{1.151}$$

We use the Pauli-Dirac representation:

$$\overrightarrow{p} = -i\hbar\overrightarrow{\nabla} \tag{1.152}$$

$$\overrightarrow{\alpha} = \begin{pmatrix} 0 & \overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{pmatrix} \tag{1.153}$$

$$\beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{1.154}$$

where $\overrightarrow{\sigma}$ is the vector of Pauli matrices and I is the 2×2 identity matrix.

We write the wave function as a two component vector of the Pauli-Dirac representation:

$$\psi(\overrightarrow{r}) = \begin{pmatrix} \varphi(\overrightarrow{r}) \\ \chi(\overrightarrow{r}) \end{pmatrix} \tag{1.155}$$

We substitute the Pauli-Dirac representation of the wave function in Dirac equation 1.151

$$c\begin{pmatrix} 0 & \overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{pmatrix}\cdot\overrightarrow{p}\begin{pmatrix} \varphi(\overrightarrow{r}) \\ \chi(\overrightarrow{r}) \end{pmatrix}+\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \varphi(\overrightarrow{r}) \\ \chi(\overrightarrow{r}) \end{pmatrix}\left(\mu c^2+S(\overrightarrow{r})\right)-\left(E-U(\overrightarrow{r})\right)\begin{pmatrix} \varphi(\overrightarrow{r}) \\ \chi(\overrightarrow{r}) \end{pmatrix}=0 \tag{1.156}$$

And we obtain two coupled differential equations:

$$c\overrightarrow{\sigma}\cdot\overrightarrow{p}\chi(\overrightarrow{r})=\left[E-U(\overrightarrow{r})-\mu c^2-S(\overrightarrow{r})\right]\varphi(\overrightarrow{r}) \tag{1.157}$$

$$c\overrightarrow{\sigma}\cdot\overrightarrow{p}\varphi(\overrightarrow{r})=\left[E-U(\overrightarrow{r})+\mu c^2+S(\overrightarrow{r})\right]\chi(\overrightarrow{r}) \tag{1.158}$$

If we consider spin symmetry, where $S(\overrightarrow{r})=U(\overrightarrow{r})$, the equation 1.157 and 1.158 become respectively:

$$c\overrightarrow{\sigma}\cdot\overrightarrow{p}\chi(\overrightarrow{r})=\left[E-2U(\overrightarrow{r})-\mu c^2\right]\varphi(\overrightarrow{r}) \tag{1.159}$$

$$c\overrightarrow{\sigma}\cdot\overrightarrow{p}\varphi(\overrightarrow{r})=\left[E+\mu c^2\right]\chi(\overrightarrow{r}) \tag{1.160}$$

Thus

$$\chi(\overrightarrow{r})=\frac{c\overrightarrow{\sigma}\cdot\overrightarrow{p}}{E+\mu c^2}\varphi(\overrightarrow{r}) \tag{1.161}$$

We substitute 1.161 in equation 1.159 we find a second order equation

$$\left[c^2p^2+2\left(E+\mu c^2\right)U(\overrightarrow{r})-\left(E^2-\mu^2c^4\right)\right]\varphi(\overrightarrow{r})=0 \tag{1.162}$$

In the same way, using pseudo-spin symmetry relation $S(\overrightarrow{r})=-U(\overrightarrow{r})$, the equations 1.157 and 1.158 become

$$c\overrightarrow{\sigma}\cdot\overrightarrow{p}\chi(\overrightarrow{r})=\left[E-\mu c^2\right]\varphi(\overrightarrow{r}) \tag{1.163}$$

$$c\overrightarrow{\sigma}\cdot\overrightarrow{p}\varphi(\overrightarrow{r})=\left[E-2U(\overrightarrow{r})+\mu c^2\right]\chi(\overrightarrow{r}) \tag{1.164}$$

Then 1.163 requires

$$\varphi(\overrightarrow{r})=\frac{c\overrightarrow{\sigma}\cdot\overrightarrow{p}}{E-\mu c^2}\chi(\overrightarrow{r}) \tag{1.165}$$

We use the last equation 1.165, the equation 1.163 gives

$$\left[c^2p^2+2\left(E+\mu c^2\right)U(\overrightarrow{r})-\left(E^2-\mu^2c^4\right)\right]\chi(\overrightarrow{r})=0 \tag{1.166}$$

We note that the two equations 1.162 and 1.166 are equivalent to the equations 1.145.

1.3.4 Solutions of Schrödinger Type Equation

The Spin Symmetry Case

The Schrödinger type equation for the spin-symmetry case is:

$$\left[c^2p^2 + 2\left(E + \mu c^2\right)U(\overrightarrow{r}) - \left(E^2 - \mu^2 c^4\right)\right]\varphi(\overrightarrow{r}) = 0 \tag{1.167}$$

With the potential energy:

$$U(\overrightarrow{r}) = \mu\left[\frac{f(\theta)}{r^2} + V(r)\right] \tag{1.168}$$

We use the polar coordinates and the same transformation as before $\psi(r,\theta) = r^{-\frac{1}{2}}R(r)\Theta(\theta)$ to get two separate equations:

To get two separate equations,the equation 1.167 becomes

$$\left[\frac{-c^2\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) + 2\left(E + \mu c^2\right)\left(\mu V(r) + \frac{\mu f(\theta)}{r^2}\right) - \left(E^2 - \mu^2 c^4\right)\right]$$
$$r^{-\frac{1}{2}}R(r)\Theta(\theta) = 0 \tag{1.169}$$

From the non-relativistic case we fond

$$\frac{\partial \psi}{\partial r} = -\frac{1}{2}r^{-\frac{3}{2}}R(r)\Theta(\theta) + r^{-\frac{1}{2}}\frac{\partial R(r)}{\partial r}\Theta(\theta) \tag{1.170}$$

$$\frac{\partial^2 \psi}{\partial r^2} = \left[\frac{3}{4}r^{-\frac{5}{2}}R(r)\Theta(\theta) - r^{-\frac{3}{2}}\frac{\partial R(r)}{\partial r}\Theta(\theta) + r^{-\frac{1}{2}}\frac{\partial^2 R(r)}{\partial r^2}\Theta(\theta)\right] \tag{1.171}$$

And

$$\frac{\partial^2 \psi}{\partial \theta^2} = r^{-\frac{1}{2}}R(r)\frac{\partial^2 \Theta(\theta)}{\partial \theta^2} \tag{1.172}$$

We substitute the derivatives in equation 1.169,thus

$$\frac{-c^2\hbar^2}{2\mu}\left[r^{-\frac{1}{2}}\frac{\partial^2 R(r)}{\partial r^2}\Theta(\theta) + \frac{1}{4}r^{-\frac{5}{2}}R(r)\Theta(\theta) + \frac{1}{r^2}r^{-\frac{1}{2}}R(r)\frac{\partial^2\Theta(\theta)}{\partial\theta^2}\right] +$$
$$\left[2\left(E + \mu c^2\right)\left(\mu V(r) + \frac{\mu f(\theta)}{r^2}\right) - \left(E^2 - \mu^2 c^4\right)\right]r^{-\frac{1}{2}}R(r)\Theta(\theta) = 0 \tag{1.173}$$

We divide by $\frac{-c^2\hbar^2}{2\mu}r^{-\frac{5}{2}}$ we find

$$\left[r^2\frac{\partial^2 R(r)}{\partial r^2}\Theta(\theta) + \frac{1}{4}R(r)\Theta(\theta)\right] + \left[-\frac{4\mu^2\left(E + \mu c^2\right)}{c^2\hbar^2}V(r) + \frac{2\mu\left(E^2 - \mu^2 c^4\right)}{c^2\hbar^2}\right]r^2 R(r)\Theta(\theta) =$$
$$-R(r)\frac{\partial^2\Theta(\theta)}{\partial\theta^2} + \frac{4\mu^2\left(E + \mu c^2\right)}{c^2\hbar^2}f(\theta)R(r)\Theta(\theta) \tag{1.174}$$

Then we divide by $R(r)\Theta(\theta)$ we get

$$\frac{1}{R(r)}\left[\left(r^2\frac{\partial^2 R(r)}{\partial r^2}+\frac{1}{4}R(r)\right)+\left(-\frac{4\mu^2\left(E+\mu c^2\right)}{\hbar^2 c^2}V(r)+\frac{2\mu\left(E^2-\mu^2 c^4\right)}{\hbar^2 c^2}\right)r^2 R(r)\right]= \frac{1}{\Theta(\theta)}\left[-\frac{\partial^2\Theta(\theta)}{\partial\theta^2}+\frac{4\mu^2\left(E+\mu c^2\right)}{\hbar^2 c^2}f(\theta)\Theta(\theta)\right] \tag{1.175}$$

We put the right part and the left part of equation 1.169 equal $-E_\theta$,we deduce two equation

$$\frac{1}{\Theta(\theta)}\left[-\frac{\partial^2\Theta(\theta)}{\partial\theta^2}+\frac{4\mu^2\left(E+\mu c^2\right)}{\hbar^2 c^2}f(\theta)\Theta(\theta)\right]=-E_\theta \tag{1.176}$$

$$\frac{1}{R(r)}\left[\left(r^2\frac{\partial^2 R(r)}{\partial r^2}+\frac{1}{4}R(r)\right)+\left(-\frac{4\mu\left(E+\mu c^2\right)}{\hbar^2 c^2}\left(\mu V(r)\right)+\frac{2\mu\left(E^2-\mu^2 c^4\right)}{\hbar^2 c^2}\right)r^2 R(r)\right]=-E_\theta \tag{1.177}$$

Thus ,the separate equations are

$$\left[\frac{d^2\Theta(\theta)}{d\theta^2}-\left(\frac{4\mu^2\left(E+\mu c^2\right)}{\hbar^2 c^2}f(\theta)+E_\theta\right)\Theta(\theta)\right]=0 \tag{1.178}$$

$$\left[\frac{d^2 R(r)}{dr^2}+\frac{1}{r^2}\left(\frac{1}{4}+E_\theta\right)R(r)+\left(-\frac{4\mu^2\left(E+\mu c^2\right)}{\hbar^2 c^2}V(r)+\frac{2\mu\left(E^2-\mu^2 c^4\right)}{\hbar^2 c^2}\right)R(r)\right]=0 \tag{1.179}$$

Now we solve this equation with the same method of the non-relativistic case

1.3.5 Relativistic Energy and Wave function (Applications)

Solution of Angular Equation The angular equation of non-relativistic case is

$$\frac{d^2\Theta(\theta)}{d\theta^2}-\left(E_\theta+\frac{2\mu^2}{\hbar^2}f(\theta)\right)\Theta(\theta)=0 \tag{1.180}$$

The angular equation of relativistic case is

$$\left[\frac{d^2\Theta(\theta)}{d\theta^2}-\left(\frac{4\mu^2\left(E+\mu c^2\right)}{\hbar^2 c^2}f(\theta)+E_\theta\right)\Theta(\theta)\right]=0 \tag{1.181}$$

We note that the angular equation of relativistic case is the same of nonrelativistic case when we put the following changes $E\longrightarrow\frac{\left(E^2-\mu^2c^4\right)}{c^2}$ and $f(\theta)\longrightarrow\frac{2\left(E+\mu c^2\right)}{c^2}f(\theta)$,so the parameters of $f(\theta)$ change from (α,β,γ) to $\left(\frac{2}{c^2}\left(E+\mu c^2\right)\alpha,\frac{2}{c^2}\left(E+\mu c^2\right)\beta,\frac{2}{c^2}\left(E+\mu c^2\right)\gamma\right)$

So the angular energy and the angular wave function of relativistic case are the same of non-relativistic case with change of the parameters

(α,β,γ) to$\left(\frac{2}{c^2}\left(E+\mu c^2\right)\alpha,\frac{2}{c^2}\left(E+\mu c^2\right)\beta,\frac{2}{c^2}\left(E+\mu c^2\right)\gamma\right)$ respectively

Solution of Radial Equation The radial equation of non-relativistic case is

$$\frac{d^2R(r)}{dr^2}+\left[\left(E_\theta+\frac{1}{4}\right)\frac{1}{r^2}-\frac{2\mu^2}{\hbar^2}V(r)+\frac{2\mu E}{\hbar^2}\right]R(r)=0 \tag{1.182}$$

The radial equation of relativistic case is

$$\left[\frac{d^2R(r)}{dr^2}+\frac{1}{r^2}\left(\frac{1}{4}+E_\theta\right)R(r)+\left(-\frac{4\mu^2\left(E+\mu c^2\right)}{\hbar^2c^2}V(r)+\frac{2\mu\left(E^2-\mu^2c^4\right)}{\hbar^2c^2}\right)R(r)\right]=0 \tag{1.183}$$

We note that the radial equation of relativistic case is the same of nonrelativistic case when we put $E\longrightarrow\frac{\left(E^2-\mu^2c^4\right)}{c^2}$ and $V(r)\longrightarrow\frac{2\left(E+\mu c^2\right)}{c^2}V(r)$,so the radial energy and the radial part of wave function of relativistic case are the same of non-relativistic case with change of the parameters

(α,β,γ) to$\left(\frac{2}{c^2}\left(E+\mu c^2\right)\alpha,\frac{2}{c^2}\left(E+\mu c^2\right)\beta,\frac{2}{c^2}\left(E+\mu c^2\right)\gamma\right)$

The Energy Spectrum and Wave Function of the System We use this transformation to write the energy and wave function of relativistic case as

For the kratzer potential

$$\frac{\left(E^2-\mu^2c^4\right)}{c^2}=-2\frac{\mu^3\left(E+\mu c^2\right)^2H^2}{c^4\hbar^2}\left(n_r+\sqrt{-E_\theta+\frac{4\mu}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r}+\frac{1}{2}\right)^{-2} \tag{1.184}$$

For the pseudoharmonic potential

$$\frac{\left(E^2-\mu^2c^4\right)}{c^2}=\hbar\sqrt{4\frac{\left(E+\mu c^2\right)}{c^2}k}\left(2n_r+1+\sqrt{-E_\theta+\frac{4\mu}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r}\right) \tag{1.185}$$

The constante of separation E_θ in relativistic case for all studied potentials , the angular part of wave function ,the radial part of relativistic wave function are shown in the $(Tables 1.8,...,1.13)$

The Pseudo-Spin Symmetry Case

The Schrödinger type equation for the pseudo-spin-symmetry case is:

$$\left[c^2p^2+2\left(E-\mu c^2\right)U(\overrightarrow{r})-\left(E^2-\mu^2c^4\right)\right]\varphi(\overrightarrow{r})=0 \tag{1.186}$$

Following the same procedure as that of spin case when just take $E-\mu c^2$ instead of $E+\mu c^2$,in this case when we take the non-relativistic limit we substitute the energy E by the non-relativistic energy $E_{n,m}=E-\mu c^2$so $E=E_{n,m}+\mu c^2$

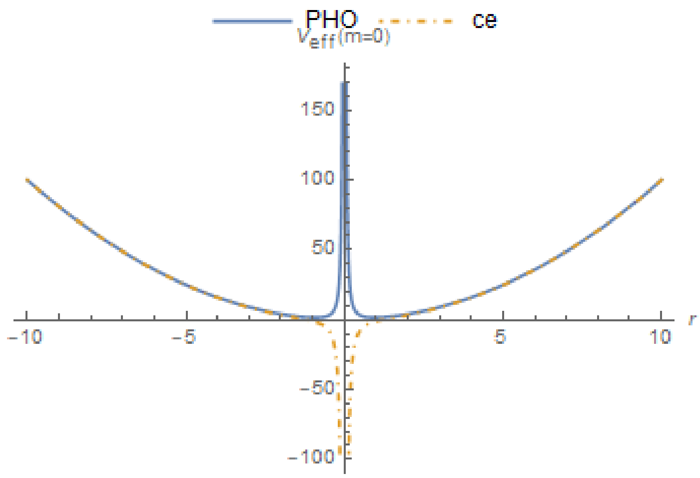

Figure 1.46: V_{eff} of (PHO+dipole) potential for $m=0, D_r=1$ and $D_\theta=2$

The Schrödinger type equation of the pseudo-spin-symmetry becomes

$$\left[c^2p^2+2\left(E_{n,m}\right)U(\overrightarrow{r})-\left(E_{n,m}^2+2E_{n,m}\mu c^2\right)\right]\varphi(\overrightarrow{r})=0 \tag{1.187}$$

The las equation can be writen as

$$\left[c^2p^2+2\left(E_{n,m}\right)U(\overrightarrow{r})-2E_{n,m}\mu c^2\left(\frac{E_{n,m}}{2\mu c^2}+1\right)\right]\varphi(\overrightarrow{r})=0 \tag{1.188}$$

We divide by $2\mu c^2$ we find

$$\left[\frac{p^2}{2\mu}+\frac{E_{n,m}}{\mu c^2}U(\overrightarrow{r})-E_{n,m}\left(\frac{E_{n,m}}{2\mu c^2}+1\right)\right]\varphi(\overrightarrow{r})=0 \tag{1.189}$$

The non-relativistic limit is obtained by neglecting the term $E_{n,m}$ beside the factor $2\mu c^2$ so we obtaine the Schrödinger equation of free partical

$$\left[\frac{p^2}{2\mu}-E_{n,m}\right]\varphi(\overrightarrow{r})=0 \tag{1.190}$$

We note that the last equation is the equation of free partical and this equation does not give us any information on potentials

The potential $V_1(r,\theta)=\mu\left[-\frac{H}{r}+\frac{D_r}{r^2}+\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos\theta)\right]$

We substitute the transformation above in the nonrelativistic energy 1.65 and wave function 1.67 we get the expression of the relativistic energy and relativistic wave function as

The relativistic energy equation is

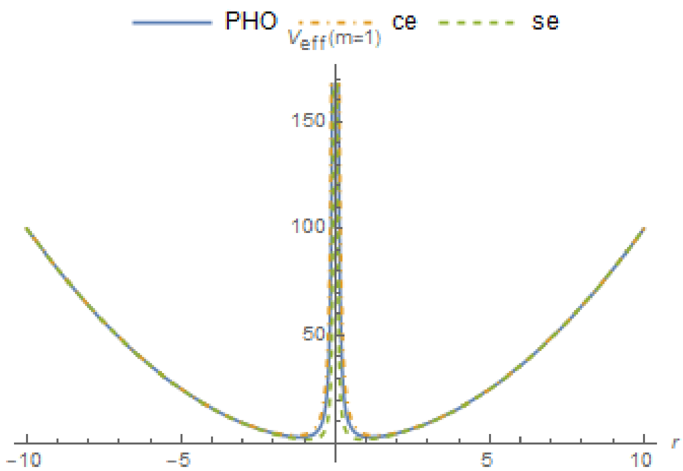

Figure 1.47: V_{eff} of (PHO+dipole) potential for $m = 1, D_r = 1$ and $D_\theta = 2$

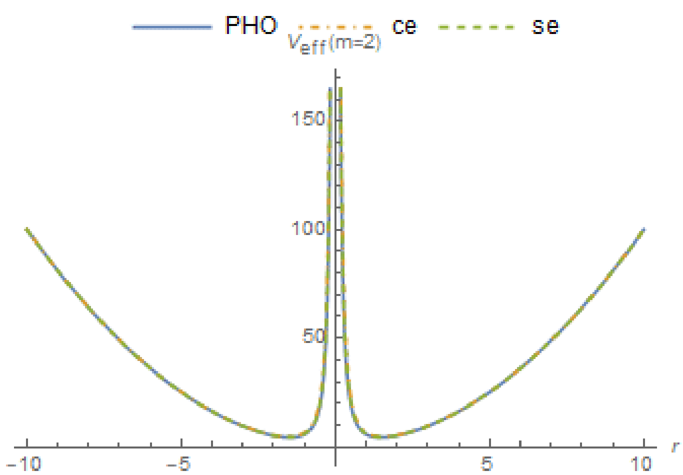

Figure 1.48: V_{eff} of (PHO+dipole) potential for $m = 2, D_r = 1$ and $D_\theta = 2$

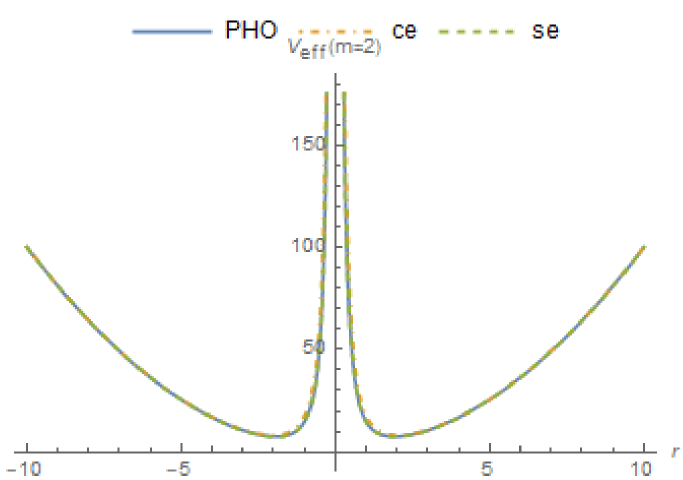

Figure 1.49: V_{eff} of (PHO+dipole) potential for $m = 2, D_r = 1$ and $D_\theta = 5$

$$\frac{\left(E^2-\mu^2c^4\right)}{c^2}=-8\frac{\mu^3\left(E+\mu c^2\right)^2H^2}{c^4\hbar^2}$$
$$\left(n_r+\sqrt{\frac{1}{4}c_{2m}\left(4\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right)+\frac{4\mu}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r}+\frac{1}{2}\right)\Bigg]^{-2} \tag{1.191}$$

$n_r=0,1,2,...$,and $m=0,1,2,...$

The relativistic wave function

$$\psi_1=Nr^{\lambda-\frac{1}{2}}e^{-\beta r}\Theta\left(\theta\right)_1F_1\left(\lambda+\frac{\mu H}{\hbar^2}\frac{2\left(E+\mu c^2\right)}{c^2}\beta^{-1},2\lambda,2\beta r\right) \tag{1.192}$$

When $\beta=\sqrt{-\frac{2m}{\hbar^2}\frac{\left(E^2-\mu^2c^4\right)}{c^2}}$

and $\lambda=\frac{1}{2}+\sqrt{\frac{1}{4}c_{2m}\left(4\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right)^2+\frac{4\mu}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r}$

The electric dipole plus kratzer potential is our contribution [74] for this reason we illustrate it in naturals units ,when we can take the following changes $H=\frac{qQ}{\pi\varepsilon_0\hbar^2}$,$D_r=\frac{qD_r}{4\pi\varepsilon_0\mu}$, $\alpha=\frac{\mu qD_\theta}{2\pi\varepsilon_0\hbar^2}$, $q=e,Q=Ze$,then the angular and radial equation becomes

$$\left[\frac{d^2\Theta(\theta)}{d\theta^2}-\left(2\frac{\left(E+\mu c^2\right)}{\hbar^2c^2}eD_\theta\cos\theta+E_\theta\right)\Theta(\theta)\right]=0 \tag{1.193}$$

$$\frac{d^2R(r)}{dr^2}+\left[\left(E_\theta-2\frac{\left(E+\mu c^2\right)}{\hbar^2c^2}eD_r+\frac{1}{4}\right)\frac{1}{r^2}+2\frac{\left(E+\mu c^2\right)}{\hbar^2c^2}Ze^2\frac{1}{r}+\frac{\left(E^2-\mu^2c^4\right)}{\hbar^2c^2}\right]R(r)=0 \tag{1.194}$$

We used the non-relativistic energies $E+\mu c^2$ and we denoted them $E_{n,m}$ the constant of separation E_θ becomes

$$E_\theta=-\frac{1}{4}c_{2m}\left(4\frac{\left(E_{n,m}+2\mu c^2\right)}{\hbar^2c^2}eD_r\right) \tag{1.195}$$

We substitute the constant of separation in the relativistic energy expression 1.191 ,we find the final expression of energy of the system as

$$\frac{\left(\left(E_{n,m}+\mu c^2\right)^2-\mu^2c^4\right)}{\hbar^2c^2}=-\left[\left(\frac{\hbar^2c^2}{\left(E_{n,m}+2\mu c^2\right)Ze^2}\right)\right.$$
$$\left(n-|m|+\sqrt{-E_\theta+2\frac{\left(E_{n,m}+2\mu c^2\right)}{\hbar^2c^2}eD_r}+\frac{1}{2}\right)\Bigg]^{-2} \tag{1.196}$$

We extract $E_\theta$from equation 1.196 we find

$$E_\theta = 2\frac{(E_{n,m}+2\mu c^2)}{\hbar^2 c^2} eD_r - \left(n - |m| + \frac{1}{2} - Z\alpha \frac{E_{n,m}+2\mu c^2}{\sqrt{\mu^2 c^4 - (E_{n,m}+\mu c^2)^2}}\right)^2 \tag{1.197}$$

Where $\alpha = -e^2$ is the fine structure constant

The non-relativistic limit is obtained by neglecting the term $E_{n,m}$ beside the factor $2\mu c^2$ in equation 1.195,

$$E_\theta = -\frac{1}{4} c_{2m} \left(8\frac{\mu}{\hbar^2} eD_r\right) \tag{1.198}$$

then we replace the last equation in 1.197 and we get the energy expression as

$$E_{n,m} = -2\mu c^2 \left(\frac{Z\alpha}{n - |m| + \sqrt{-E_\theta + 4\frac{\mu}{\hbar^2} eD_r} + \frac{1}{2}}\right)^2 \tag{1.199}$$

use the Taylor series according to α^2:

$$E_{n,m} = -\frac{8\mu c^2 Z\alpha^2}{\left(n - |m| + \sqrt{-E_\theta + 4\frac{\mu}{\hbar^2} eD_r} + \frac{1}{2}\right)^2} + \frac{32\mu c^2 Z\alpha^4}{\left(n - |m| + \sqrt{-E_\theta + 4\frac{\mu}{\hbar^2} eD_r} + \frac{1}{2}\right)^2} + O\left(\alpha^6\right) \tag{1.200}$$

We use the Hartree units ($\hbar = e = \mu = 4\pi\varepsilon_0 = 1$) for the numerical computations the equations 1.197 and 1.195 become :

$$E_\theta = -\frac{1}{4} c_{2m} \left(4(E_{n,m}\alpha^2 + 2)D_\theta\right) \tag{1.201}$$

$$E_\theta = 2(E_{n,m}\alpha^2 + 2)D_r - \left(n + |m| - \frac{1}{2} - Z\alpha \frac{E_{n,m}\alpha^2 + 2}{\sqrt{1 - (E_{n,m}\alpha^2 + 1)^2}}\right)^2 \tag{1.202}$$

And the non-relativistic limit becomes($Z = 1$):

$$E_{n,m} = -\frac{2}{\left(n - |m| + \sqrt{\frac{1}{4} c_{2m} \left(8D_r\right) + 4D_r} + \frac{1}{2}\right)^2} + \frac{8\alpha^2}{\left(n - |m| + \sqrt{\frac{1}{4} c_{2m} \left(8D_r\right)} + \frac{1}{2}\right)^2} + O\left(\alpha^6\right) \tag{1.203}$$

We see here that 1.203 differs from 1.74by a factor of 2 in front of the dipoles moments D_θ and D_r. This factor comes from the addition of scalar and vector potentials in spin-symmetry case which gives a Schrödinger equation with a potential $2V$ instead of V in ordinary theory [100],[101] . We cannot solve the system of equations 1.201 and 1.202 analytically because

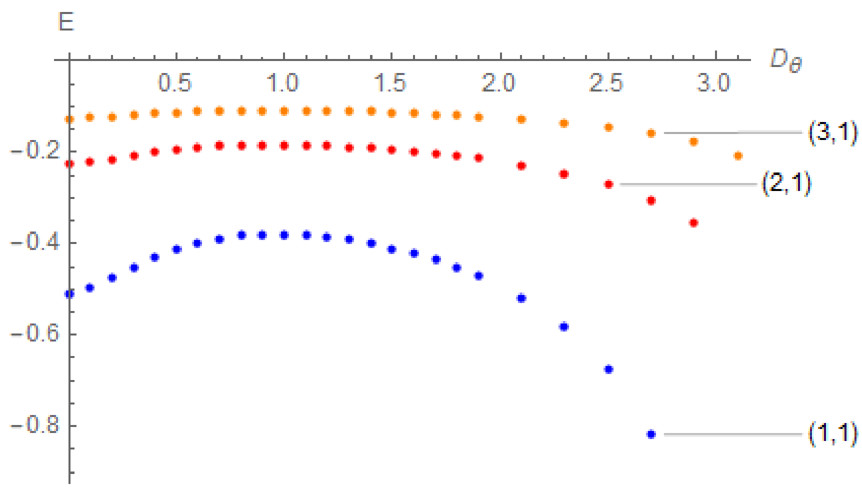

Figure 1.50: Relativistic energy $E(n,1)$ in terms of D_θ for $D_r = 0.3$ and $n = 1,2$ and 3

Mathieu characteristics don't have inverse functions. Nevertheless, this system can be solved using graphical methods by seeking the intersection points of the graphs representing the two equations.equation 1.202 shows that $E_\theta$has an inverted and non-symmetric parabolic shape and the intersection point with the plots representing 1.201 can not exceed its maximum; This limitation gives the critical dipole moments for each quantum numbers. Unlike the non-relativistic case where D_{crit} depends only on the value of m, its values here are weakly dependent on the other quantum number n. This dependence on n comes from the presence of the energies En,m with D in the angular eigenvalues 1.201 and these energies depend on n as can be seen from 1.202. The weakness of this dependence comes from the presence of the factor α^2 with $E_{n,m}$. The study of the dependence of the energies according to the values of D_θ shows that this moment increases the energies of the system to a maximum value and then its effect is transformed into a decrease thereof; This shape follows that of the c_{2m} and it is common to all levels but decreases with increasing n. The effect of $D_r$can be summarized in a shift of the energies to larger or smaller values depending on its sign $(Figures 1.50\ and\ 1.51)$. We mention here that the non-relativistic approximation 1.203 can be used as a quasi-analytical solution since it gives results in excellent agreement with those computed numerically $(Figure\ 1.52)$.

For the pseudo-spin symmetry case following the same procedure as that of the spin case, we end up with two relations that come from the eigenvalues of radial and angular equations. We find the following relations for the nonrelativistic energies of the system $E_{n,m} = E - \mu c^2$ (In Hartree units):

$$E_\theta = -\frac{1}{4} c_{2m} \left(4E_{n,m}\alpha^2 D_\theta\right) \tag{1.204}$$

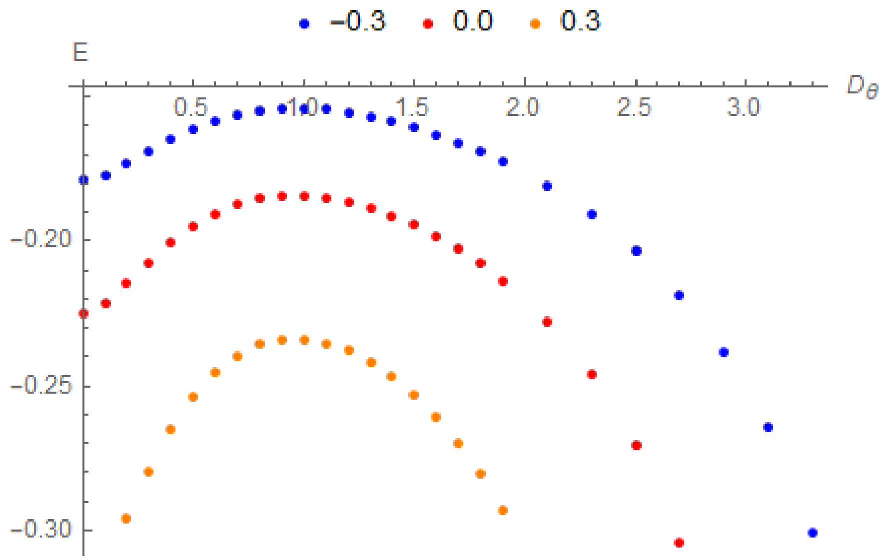

Figure 1.51: $E(2,1)$ in terms of D_θ for $D_r = 0, 0.3$ and 0.6

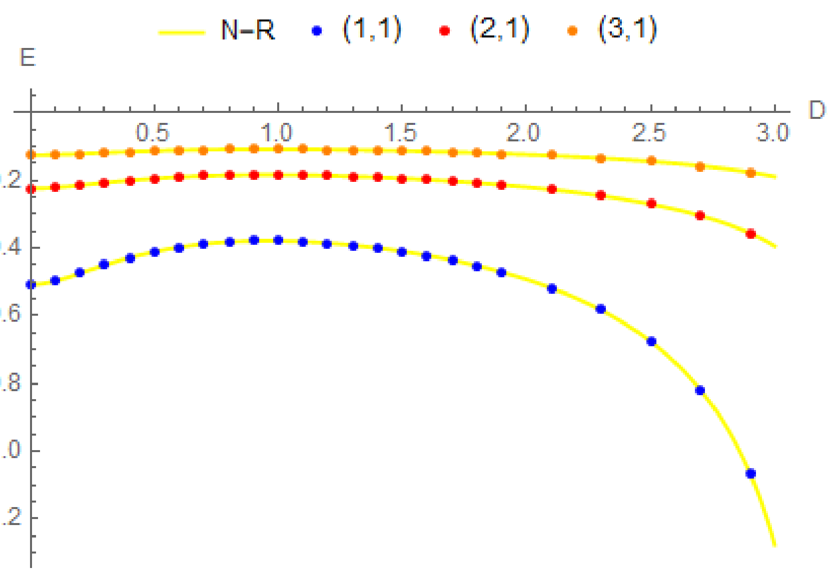

Figure 1.52: Relativistic and Non-Relativistic energy $E(1,1), E(2,1)$ and $E(3,1)$ in terms of D_θ for $D_r = 0.3$

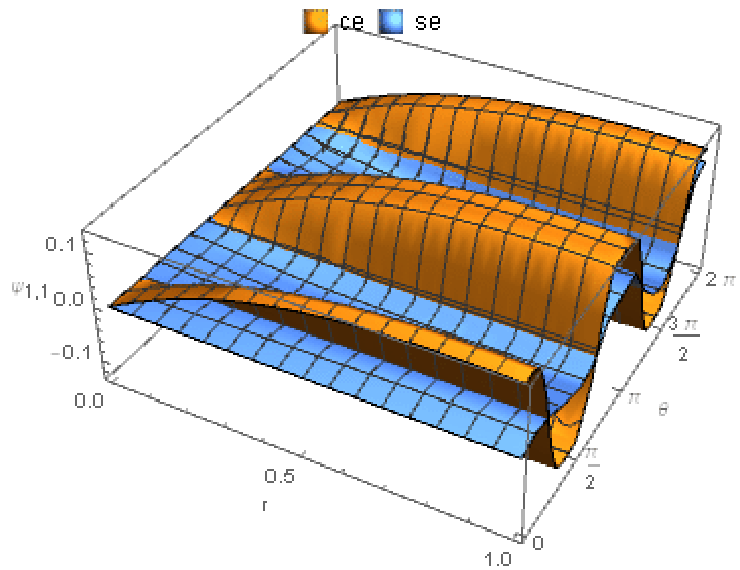

Figure 1.53: The wave function $\psi(r,\theta)$ in terms of D_θ and D_r

$$E_\theta = 2E_{n,m}\alpha^2 D_r - \left(n - |m| - \frac{1}{2} - Z\alpha\frac{E_{n,m}\alpha^2}{\sqrt{1-(E_{n,m}\alpha^2+1)^2}}\right)^2 \qquad (1.205)$$

The main difference between these equations and those of the spin symmetry case (1.2011.202) is the absence of factor 2 in front of the α^2term. This means that the graphs representing the radial solution 1.205 are almost linear and that the parameter p inside the Mathieu characteristics 1.204 is very small. Our calculations show that we have to consider very large radial moments ($Dr > 100$ $a.u.$) to find solutions higher than -200 $a.u.$. These results are outside the regions of interest for the energies of the atomic systems and support those of works that consider only the case of spin symmetry in their studies,[101],[102] .

In the non-relativistic case, the spectrum shows that the energies follow mainly the behavior of Mathieu's characteristic parameters and thus the angular moment D_θ, whereas the effect of the radial moment D_r is merely a shift in these energies to larger or smaller values according to its sign. We have shown also that there is an essential condition for bound states to exist, which is: $c_{2m}(4D_\theta) + 8D_r > 0$. This condition imposes a critical value for the angular moment D_θ, depending on the value of m, otherwise the corresponding bound state disappears. These critical values of D_θ depend also on the value of D_r and the negative value of this moment which makes $c_{2m}(4D_\theta) + 8D_r = 0$ is also a critical value for the radial moment. So we see that by increasing, the radial dipole displaces the energies towards larger values while widening the region of possible values of the angular moment. In the relativistic cases the eigenfunctions are determined analytically but the energies can only be calculated using graphical methods. Only the spin symmetry has given results corresponding to atomic systems. The behavior of the energies is the same as that of the Schrödinger spectrum but it is shifted because the Schrödinger type equation of the relativistic systems has $2V$ as a potential instead of the potential V in the ordinary Schrödinger equation. We also note that the critical values of the dipole moments D_r and D_θ depend on the two quantum numbers n and m in the relativistic case instead of just m in the case of non-relativistic systems. We have found that the angular term removes the degeneracy found in the $exp(im\theta)$ part of the solutions for central potentials. This is equivalent to the effect of a constant magnetic field in $3D$ systems, where its action removes the degeneracy of the $exp(im\phi)$ solutions too. In both cases, the privileged direction of the interaction (dipole axis in $2D$ and field direction in $3D$) removes the degeneracy that existed due to the isotropy of the action before.

For the potential $\mathbf{V}_2(r,\theta) = \frac{\mu}{q}\left[-\frac{H}{r} + \left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_1(r,\theta)$ when we put $D_r \longrightarrow 0$ so

$$\frac{\left(E^2-\mu^2c^4\right)}{c^2} = --8\frac{\mu^3\left(E+\mu c^2\right)^2 H^2}{c^4\hbar^2}\left[\left(n_r + \sqrt{\frac{1}{4}c_{2m}\left(4\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right)} + \frac{1}{2}\right)\right]^{-2} \tag{1.206}$$

The relativistic wave function

$$\psi_2 = N r^{\lambda-\frac{1}{2}} e^{-\beta r}\Theta(\theta)\ {}_1F_1\left(\lambda + \frac{\mu H}{\hbar^2}\frac{2\left(E+\mu c^2\right)}{c^2}\beta^{-1}, 2\lambda, 2\beta r\right) \tag{1.207}$$

When $\beta = \sqrt{-\frac{2m}{\hbar^2}\frac{(E^2-\mu^2c^4)}{c^2}}$

and$\lambda = \frac{1}{2} + \sqrt{-\frac{2(E+\mu c^2)}{c^2}(\alpha-\gamma) + \frac{\left[m+\frac{1}{2}+\frac{1}{2}\left(1+16\frac{2(E+\mu c^2)}{c^2}\alpha\right)^{1/2}\right]-4\beta^2}{4\left[m+\frac{1}{2}+\frac{1}{2}\left(1+16\frac{2(E+\mu c^2)}{c^2}\alpha\right)^{1/2}\right]^2}}^{\,2}$

The potential $V_3(r,\theta) = \mu\left[kr^2 + \frac{D_r}{r^2} + \left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta\right]$

We substitute the transformation above in the nonrelativistic energy 1.103 and wave function 1.108 we get the expression of the relativistic energy and relativistic wave function as

The relativistic energy equation is

$$\left(E-\mu c^2\right)\sqrt{\left(E+\mu c^2\right)} = 2\hbar c\sqrt{k}\left[n - |m| + 1 + \sqrt{\frac{1}{4}c_{2m}\left(4\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right) + \frac{4\mu}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r}\right] \tag{1.208}$$

We substitute by the relation $E_{n,m} + \mu c^2 = E$,so

$$E_{n,m}\sqrt{\left(E_{n,m}+2\mu c^2\right)}=2\hbar c\sqrt{k}\,[n-|m|+1+$$
$$\sqrt{\frac{1}{4}c_{2m}\left(4\frac{\left(E_{n,m}+2\mu c^2\right)}{c^2}\alpha\right)+\frac{4\mu}{\hbar^2}\frac{\left(E_{n,m}+2\mu c^2\right)}{c^2}D_r}\Bigg] \tag{1.209}$$

by neglecting the term $E_{n,m}$ beside the factor $2\mu c^2$

$$E_{n,m}=\hbar\sqrt{\frac{2k}{\mu}}\left[n-|m|+1+\sqrt{\frac{1}{4}c_{2m}\left(8\mu\alpha\right)+\frac{8\mu^2}{\hbar^2}D_r}\right] \tag{1.210}$$

In the Hartree units system and

$$E_{n,m}=\left[n-|m|+1+\sqrt{\frac{1}{4}c_{2m}\left(16D_\theta\right)+8D_r}\right] \tag{1.211}$$

This expression of the non-relativistic energy of the spin symmetry case is defferent by the number 4 for the contribution of the dipole and Kratzer potential ,the variation of this energy in terms of D_θ and D_r is showen in $(Figures\ 1.27, 1.28)$

The wave function is

$$\psi_3=N\frac{(r)^{2\alpha-\frac{1}{2}}}{(a)^{2\alpha}}e^{-\frac{r^2}{2a^2}}\Theta(\theta)_1F_1\left(\left(\alpha+\frac{1}{4}\right)-\frac{\varepsilon a^2}{4},2\alpha+\frac{1}{2},\frac{r^2}{a^2}\right) \tag{1.212}$$

When $a^2=\sqrt{\frac{\hbar^2}{2\mu\frac{2(E+\mu c^2)}{c^2}k}}$,$\varepsilon=\frac{2\mu}{\hbar^2}\frac{\left(E^2-\mu^2c^4\right)}{c^2}$,

$\alpha=\frac{1}{2}\left(\frac{1}{2}+\sqrt{1-4\eta}\right)$and

$$\eta=\left(\frac{2\left(E+\mu c^2\right)}{c^2}\left(\alpha-\gamma\right)-\frac{\left[m+\frac{1}{2}+\frac{1}{2}\left(1+16\frac{2\left(E+\mu c^2\right)}{c^2}\alpha\right)^{1/2}\right]-4\beta^2}{4\left[m+\frac{1}{2}+\frac{1}{2}\left(1+16\frac{2\left(E+\mu c^2\right)}{c^2}\alpha\right)^{1/2}\right]^2}+\frac{1}{4}-\frac{2\mu}{\hbar^2}\frac{2\left(E+\mu c^2\right)}{c^2}D_r\right)$$

For the potential $\mathbf{V}_4\left(r,\theta\right)=\boldsymbol{\mu}\left[kr^2+\left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_3\left(r,\theta\right)$ when we put $D_r\longrightarrow 0$ so the relativistic energy equation is

$$\left(E-\mu c^2\right)\sqrt{\left(E+\mu c^2\right)}=\hbar 2c\sqrt{k}\left[n-|m|+1+\sqrt{\frac{1}{4}c_{2m}\left(4\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right)}\right] \tag{1.213}$$

$n=0,1,2,...,$ and $m=0,1,2,...$

The relativistic wave function

$$\psi_4=N\frac{(r)^{2\alpha-\frac{1}{2}}}{(a)^{2\alpha}}e^{-\frac{r^2}{2a^2}}\Theta(\theta)_1F_1\left(\left(\alpha+\frac{1}{4}\right)-\frac{\varepsilon a^2}{4},2\alpha+\frac{1}{2},\frac{r^2}{a^2}\right) \tag{1.214}$$

When $a^2 = \sqrt{\frac{\hbar^2}{2\mu \frac{2(E+\mu c^2)}{c^2} k}}$,$\varepsilon = \frac{2\mu}{\hbar^2} \frac{\left(E^2-\mu^2 c^4\right)}{c^2}$,

$\alpha = \frac{1}{2}\left(\frac{1}{2} + \sqrt{1-4\eta}\right)$ and

$$\eta = \left(\frac{2\left(E+\mu c^2\right)}{c^2}(\alpha-\gamma) - \frac{\left[m+\frac{1}{2}+\frac{1}{2}\left(1+16\frac{2\left(E+\mu c^2\right)}{c^2}\alpha\right)^{1/2}\right]-4\beta^2}{4\left[m+\frac{1}{2}+\frac{1}{2}\left(1+16\frac{2\left(E+\mu c^2\right)}{c^2}\alpha\right)^{1/2}\right]^2} + \frac{1}{4} \right) \frac{\left(\alpha \sin^2\theta + \beta\sin\theta + \gamma\right)}{\cos^2}$$

The results of the 2D relativistic studies is summarized in the $(Tables 1.8, ..., 1.13)$

1.4 Discussion

In this chapter, we studied some non-central potentials $V(r,\theta) = \mu\left[V(r) + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)f(\theta)\right]$ for $2D$ quantum systems in both non-relativistic and relativistic cases. We solved the Schrödinger equation analytically and studied the relativistic spectrum for Klein-Gordon and Dirac equations in both spin and pseudo-spin symmetry We note in this chapter that in the $2D$ space to find a bond state of a particle moving in noncentral potential and with the presence of kratzer or pseudoharmonic potential the following condition $\frac{2\mu^2}{\hbar^2}D_r - E_\theta \geq 0$ must be fulfilled this give as a critical values for the parameters of the noncentral potential and this critical value influenced by the parameters of the kratzer potential when it can get bigger or smaller.Unlike other potentials, the dipole + Kratzer potential $\mu\left[-\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta\right]$, gave good results .when in the non-relativistic case, the spectrum shows that the energies follow mainly the behavior of Mathieu's characteristic parameters and thus the angular moment $D_\theta = \frac{\alpha}{2}$, whereas the effect of the radial moment D_r is merely a shift in these energies to larger or smaller values according to its sign. We have showed also that there is an essential condition for bound states to exist, which is: $c_{2m}(4D_\theta) + 8D_r > 0$. This condition imposes a critical value for the angular moment D_θ,depending on the value of m, otherwise the corresponding bound state disappears. These critical values of D_θ depend also on the value of D_r and the negative value of this moment which makes $c_{2m}(4D_\theta) + 8D_r = 0$ is also a critical value for the radial moment. So we see that by increasing the radial dipole displaces the energies towards the larger values while widening the region of the possible values of the angular moment.in the relativistic cases the eigenfunctions are determined analytically but the energies can only be calculated using graphical methods. Only the spin symmetry has given results corresponding to atomic systems. The behavior of the energies is the same as that of the Schr¨odinger spectrum but it is shifted because the Schrödinger type equation of the relativistic systems has $2V$ as a potential instead of the potential V in the ordinary Schr¨odinger equation. We also note that the critical values of the dipole moments D_r and D_θ depend on the two quantum numbers n and m in the relativistic case instead of just m in the case of non-relativistic systems.We have found that the angular term removes the degeneracy found in the $exp(im\theta)$ part of the solutions for central potentials. This is equivalent to the effect of a constant magnetic field in $3D$ systems, where its action removes the degeneracy of the

$f(\theta)$	E_θ
$\left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta$	$-\frac{1}{4}c_{2m}\left(4\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right)$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\frac{\left(\alpha\sin^2\theta+\beta\sin\theta+\gamma\right)}{\cos^2}$	$\frac{2\left(E+\mu c^2\right)}{c^2}\alpha-$ $\left[m+\frac{1}{2}+\frac{1}{4}(1+\frac{8\left(E+\mu c^2\right)}{c^2}(\alpha+\beta+\gamma))^{1/2}+\frac{1}{4}(1+\frac{8\left(E+\mu c^2\right)}{c^2}(\alpha-\beta+\gamma))^2\right]$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\left(\alpha\tan^2\frac{\theta}{2}+\beta\tan\frac{\theta}{2}+\gamma\right)$	$\frac{2\left(E+\mu c^2\right)}{c^2}(\alpha-\gamma)-$ $\frac{\left[m+\frac{1}{2}+\frac{1}{2}\left(1+32\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right)^{1/2}\right]-16\left(\frac{\left(E+\mu c^2\right)}{c^2}\beta\right)^2}{4\left[m+\frac{1}{2}+\frac{1}{2}\left(1+32\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right)\right]^2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\left(\alpha\cot^2\frac{\theta}{2}+\beta\cot\frac{\theta}{2}+\gamma\right)$	$\frac{2\left(E+\mu c^2\right)}{c^2}(\alpha-\gamma)-$ $\frac{\left[m+\frac{1}{2}+\frac{1}{2}\left(1+32\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right)^{1/2}\right]-16\left(\frac{\left(E+\mu c^2\right)}{c^2}\beta\right)^2}{4\left[m+\frac{1}{2}+\frac{1}{2}\left(1+32\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right)\right]^2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $(\alpha\tan^2\theta+\beta\tan\theta+\gamma)$	$\frac{2\left(E+\mu c^2\right)}{c^2}(\alpha-\gamma)-$ $\frac{\left[\left(1+8\frac{\left(E+\mu c^2\right)}{c^2}\alpha\right)^{1/2}+1+2m\right]^4-16\left(\frac{\left(E+\mu c^2\right)}{c^2}\beta\right)^2}{4\left[\left(1+8\frac{\left(E+\mu c^2\right)}{c^2}\right)^{1/2}+1+2m\right]^2}$

Table 1.8: The relativistic 2D constant of separation

$f(\theta)$	$\Theta(\theta)$
$\left(\frac{\hbar^2}{2\mu^2}\right)\alpha\cos\theta$	$Mathieu function$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\sin^2\theta+\beta\sin\theta+\gamma\right)\cos^{-2}$	$\left(\frac{1-\sin\theta}{2}\right)^{\rho}\left(\frac{1+\sin\theta}{2}\right)^{\sigma}F\left(2\rho,2\sigma,(2\rho+\frac{1}{2});\frac{1-\sin\theta}{2}\right)$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\frac{\theta}{2}+\beta\tan\frac{\theta}{2}+\gamma\right)$	$-e^{i\rho\theta}(1+e^{i\theta})^{\sigma}F\left(2\rho,2\sigma,(2\rho+1);-e^{i\theta}\right)$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cot^2\frac{\theta}{2}+\beta\cot\frac{\theta}{2}+\gamma\right)$	$(-1)^{i\rho+1}\,e^{i\rho\theta}(1-e^{i\theta})^{\sigma}F\left(2\rho,2\sigma,(2\rho+1);e^{i\theta}\right)$
$\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\theta+\beta\tan\theta+\gamma\right)$	$\left(1+e^{2i\theta\rho}\right)(-e^{2i\theta})^{\sigma}F\left(2\rho,2\sigma,1+(1+4\alpha)^{1/2};1+e^{2i\theta}\right)$

Table 1.9: The relativistic 2D angular part of wave function

$f(\theta)$	ρ	σ
Case 2	$\frac{1}{4}+\frac{1}{4}(1+8\frac{(E+\mu c^2)}{c^2}(\alpha+\beta+\gamma))^{1/2}$	$\frac{1}{4}+\frac{1}{4}(1+8\frac{(E+\mu c^2)}{c^2}(\alpha-\beta+\gamma))^{1/2}$
Case 3	$(-E_\theta+2\frac{(E+\mu c^2)}{c^2}(\alpha-i\beta-\gamma))^{1/2}$	$\frac{1}{2}+\frac{1}{2}(1+32\frac{(E+\mu c^2)}{c^2}\alpha)^{1/2}$
Case 4	$(-E_\theta+2\frac{(E+\mu c^2)}{c^2}(\alpha-i\beta-\gamma))^{1/2}$	$\frac{1}{2}+\frac{1}{2}(1+32\frac{(E+\mu c^2)}{c^2}\alpha)^{1/2}$
Case 5	$\frac{1}{2}+\frac{1}{2}(1+8\frac{(E+\mu c^2)}{c^2})^{1/2}$	$\frac{1}{2}(-E_\theta+2\frac{(E+\mu c^2)}{c^2}(\alpha-i\beta-\gamma))^{1/2}$

Table 1.10: The parameters of the relativistic 2D constant of separation

$V(r)$	$R(r)$	λ	β^2
$-\frac{H}{r}+\frac{D_r}{r^2}$	$N_r r^{\lambda} e^{-\beta r}\ {}_1F_1\left(-n_r, 2\lambda, 2\beta r\right)$	$\frac{1}{2}+\sqrt{-E_\theta+\frac{4\left(E+\mu c^2\right)\mu^2 D_r}{\hbar^2 c^2}}$	$-\frac{2\mu}{\hbar^2}\frac{\left(E^2-\mu^2 c^4\right)}{c^2}$
$-\frac{H}{r}$	$N_r r^{\lambda} e^{-\beta r}\ {}_1F_1\left(-n_r, 2\lambda, 2\beta r\right)$	$\frac{1}{2}+\sqrt{-E_\theta}$	$-\frac{2\mu}{\hbar^2}\frac{\left(E^2-\mu^2 c^4\right)}{c^2}$
$kr^2+\frac{D_r}{r^2}$	$N_r\left(\frac{r}{\beta}\right)^{\frac{1}{2}+\frac{\sqrt{1-4\lambda}}{2}} e^{-\frac{r^2}{2\beta^2}}$ ${}_1F_1\left(-n_r, 1+\frac{\sqrt{1-4\lambda}}{2}, \frac{r^2}{\beta^2}\right)$	$E_\theta+\frac{1}{4}-\frac{4\left(E+\mu c^2\right)\mu^2 D_r}{\hbar^2 c^2}$	$\frac{\hbar}{\mu\sqrt{4\frac{\left(E+\mu c^2\right)}{c^2}k}}$
kr^2	$N_r\left(\frac{r}{\beta}\right)^{\frac{1}{2}+\frac{\sqrt{1-4\lambda}}{2}} e^{-\frac{r^2}{2\beta^2}}$ ${}_1F_1\left(-n_r, 1+\frac{\sqrt{1-4\lambda}}{2}, \frac{r^2}{\beta^2}\right)$	$E_\theta+\frac{1}{4}$	$\frac{\hbar}{\mu\sqrt{4\frac{\left(E+\mu c^2\right)}{c^2}k}}$

Table 1.11: The relativistic 2D radial part of wave function

$V(r,\theta)$	$\frac{\left(E^2-\mu^2c^4\right)}{c^2}$
$\mu\left(-\frac{H}{r}+\frac{D_r}{r^2}+\frac{f\left(\theta\right)}{r^2}\right)$	$-8\frac{\mu^3\left(E+\mu c^2\right)^2H^2}{c^4\hbar^2}\left[\left(n_r+\sqrt{-E_\theta+\frac{4\mu}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r}+\frac{1}{2}\right)\right]^{-2}$
$\mu\left(-\frac{H}{r}+\frac{f\left(\theta\right)}{r^2}\right)$	$-8\frac{\mu^3\left(E+\mu c^2\right)^2H^2}{c^4\hbar^2}\left[\left(n_r+\sqrt{-E_\theta}+\frac{1}{2}\right)\right]^{-2}$
$\mu\left(kr^2+\frac{D_r}{r^2}+\frac{f\left(\theta\right)}{r^2}\right)$	$\hbar\sqrt{4\frac{\left(E+\mu c^2\right)}{c^2}k}\left[2n_r+1+\sqrt{-E_\theta+\frac{4\mu}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r}\right]$
$\mu\left(kr^2+\frac{f\left(\theta\right)}{r^2}\right)$	$\hbar\sqrt{4\frac{\left(E+\mu c^2\right)}{c^2}k}\left[2n_r+1+\sqrt{-E_\theta}\right]$

Table 1.12: Equation of 2D relativistic energy

$V(r,\theta)$	$\psi(r,\theta)$
$\mu\left(-\frac{H}{r}+\frac{D_r}{r^2}+\frac{f\left(\theta\right)}{r^2}\right)$	$Nr^{\lambda-\frac{1}{2}}e^{-\beta r}\ {}_1F_1\left(-n_r,2\lambda,2\beta r\right)\Theta\left(\theta\right)$
$\mu\left(-\frac{H}{r}+\frac{f\left(\theta\right)}{r^2}\right)$	$Nr^{\lambda-\frac{1}{2}}e^{-\beta r}\ {}_1F_1\left(-n_r,2\lambda,2\beta r\right)\Theta\left(\theta\right)$
$\mu\left(kr^2+\frac{D_r}{r^2}+\frac{f\left(\theta\right)}{r^2}\right)$	$Nr^{-\frac{1}{2}}\left(\frac{r}{\beta}\right)^{\frac{1}{2}+\frac{\sqrt{1-4\lambda}}{2}}e^{-\frac{r^2}{2\beta^2}}{}_1F_1\left(-n_r,1+\frac{\sqrt{1-4\lambda}}{2},\frac{r^2}{\beta^2}\right)\Theta\left(\theta\right)$
$\mu\left(kr^2+\frac{f\left(\theta\right)}{r^2}\right)$	$Nr^{-\frac{1}{2}}\left(\frac{r}{\beta}\right)^{\frac{1}{2}+\frac{\sqrt{1-4\lambda}}{2}}e^{-\frac{r^2}{2\beta^2}}{}_1F_1\left(-n_r,1+\frac{\sqrt{1-4\lambda}}{2},\frac{r^2}{\beta^2}\right)\Theta\left(\theta\right)$

Table 1.13: Relativistic 2D wave function

$exp(im\phi)$ solutions too. In both cases,the privileged direction of the interaction (dipole axis in $2D$ and field direction in $3D$) removes the degeneracy that existed due to the isotropy of the action before

Also we studied a systemof quantum ring confined by apseudoharmonic potential and under the effect of a dipolar impurity and we find that The first characteristic of the dipole term is that it removes the degeneracy present for central potentials; thus the energies depend on the orientation of the solutions compared to the dipole direction, which broke the central symmetry by becoming a privileged one. Corrections are more pronounced for ce states and therefore states whose orientations are in the same direction as the dipole; this is similar to the dependence of $3D$ energies on the azimuth number m as soon as we are in the presence of a Hamiltonian term depending on the direction like a constant magnetic field.Our solutions generalize the azimuthal quantum number m through the Mathieu characteristic values. The corrections are larger for $m = 0$ and they decrease as it increases; this generates a correction on the transition energies between the different levels and it is more apparent for those between the lowest ones as $(n, 1) \longrightarrow (n, 0)$ and $(n, 2) \longrightarrow (n, 1)$. All these corrections depends on the chosen materia.Regarding to the relativistic study of pseudoharmonic dipole we find the relativistic energy take the same non-relative energy curve but with a shift in all levels

Chapter 2

Studies of Three Dimensional Non-Central Potentials

2.1 Introduction

counter to two dimensional quantum mechanic the three dimensional quantum mechanics have used extensively in all fields of science particularly in chemistry and also in nuclear physics when the non-central potential give arises as a good description of a ro-vibrational energy levels of the molecules, atoms, and distorted nucleus In recent years many efforts have been made to solve the Schrödinger equation for non-central potentials in three dimensions like Hartmann potential,The non-central Makarov potential,the Coulombic ring-shaped potential,deformed ring-shaped potential,,double ring-shaped Coulomb potential and this potentials is a limits of a non-central potentials of Hautot which mentioned in $(Table\ 2)$ On the other hand, to study these potentials in the relativistic case, and with the difficulty of solving the Dirac and Klein Gordon equation, many researchers have resorted to the use of spin and pseaudospin symmetry

This chapter is arranged as follows: in section 2,we focused to the nonrelativistic case when we write the Schrödinger equation in spherical coordinates for a particle in the presence of non-central potential and separated it into radial and angular parts, we solve this separate equations to get the nonrelativistic energy and the nonrelativistic wave function In section 3 we illustrate the spin symmetry and pseaudospin symmetry limit of relativistic case when we deduced the relativistic energy and the relativistic wave function,also in this chapter we focused extensively on ring-shaped potential where we plotted its energy and we have discussed its variations ,The studied potentials in this chapter are shown in graphs $(Figures\ 2.1, ..., 2.12)$

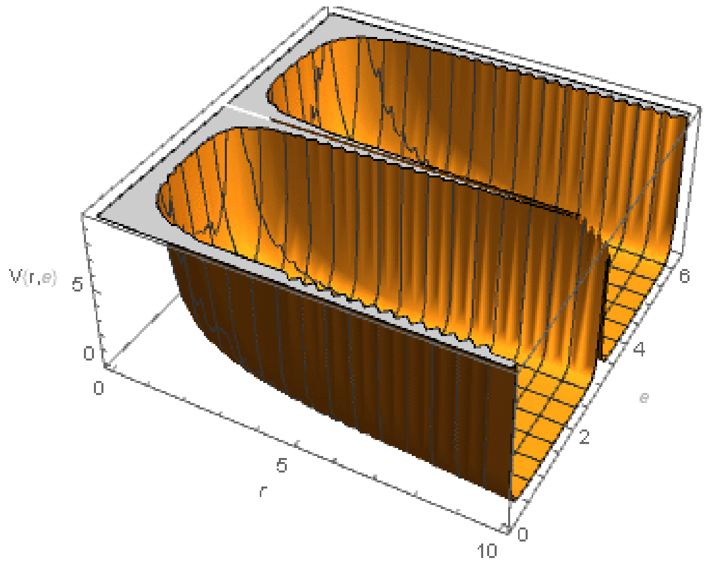

Figure 2.1: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta + \beta\cos\theta + \gamma)\sin^{-2}\theta$ in terms of r and θ

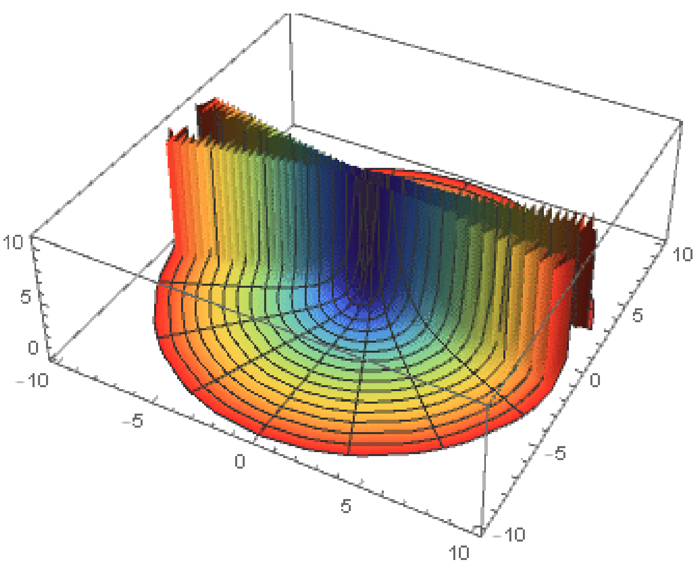

Figure 2.2: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta + \beta\cos\theta + \gamma)\sin^{-2}\theta$ in terms of r and θ in cylindrical coordinates system

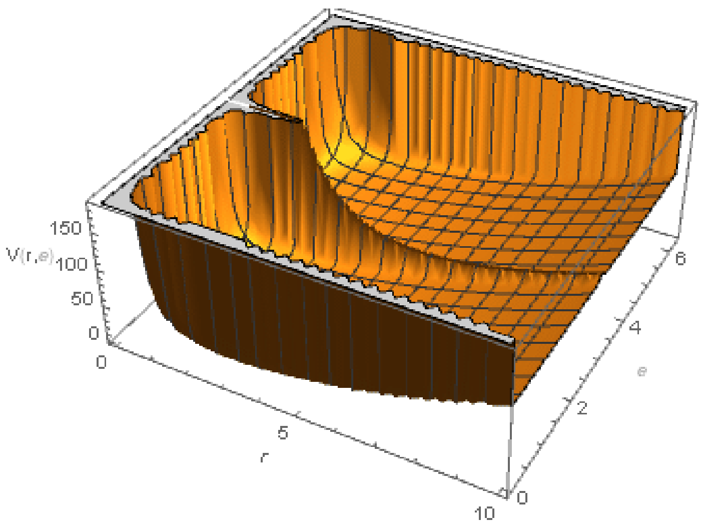

Figure 2.3: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta + \beta\cos\theta + \gamma)\sin^{-2}\theta$ in terms of r and θ

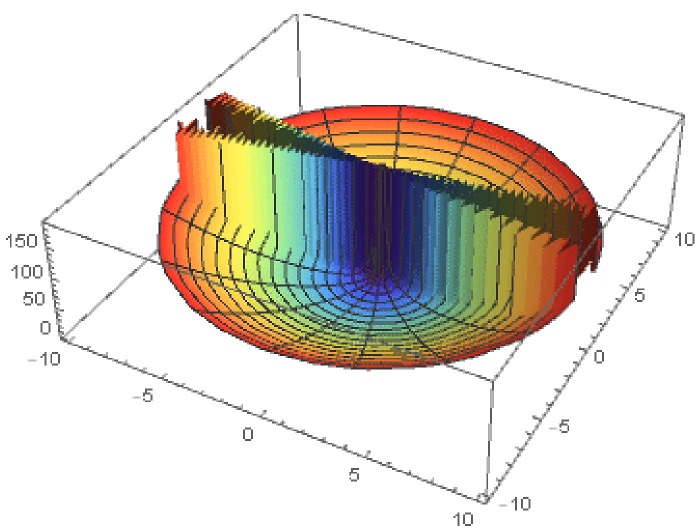

Figure 2.4: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta + \beta\cos\theta + \gamma)\sin^{-2}\theta$ in terms of r and θ in cylindrical coordinates system

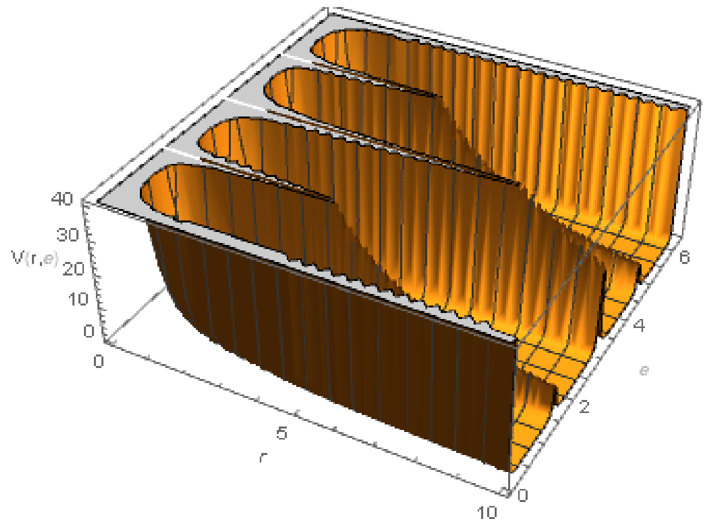

Figure 2.5: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^4\theta + \beta\cos^2\theta + \gamma)\sin^{-2}\theta\cos^{-2}\theta$ in terms of r and θ

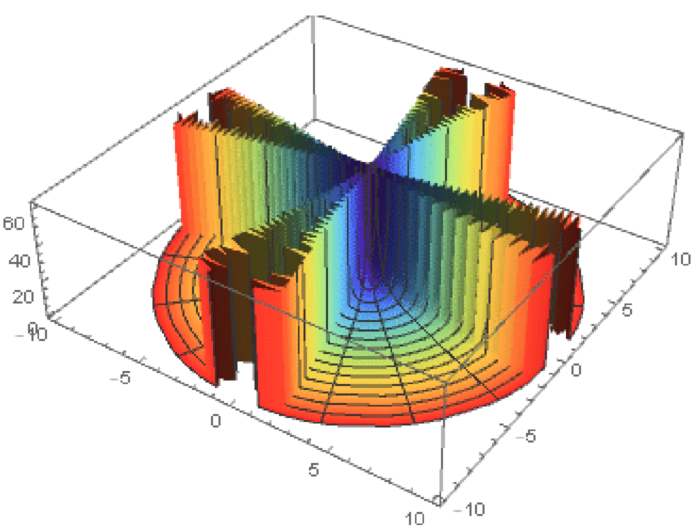

Figure 2.6: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^4\theta + \beta\cos^2\theta + \gamma)\sin^{-2}\theta\cos^{-2}\theta$ in terms of r and θ in cylindrical coordinates system

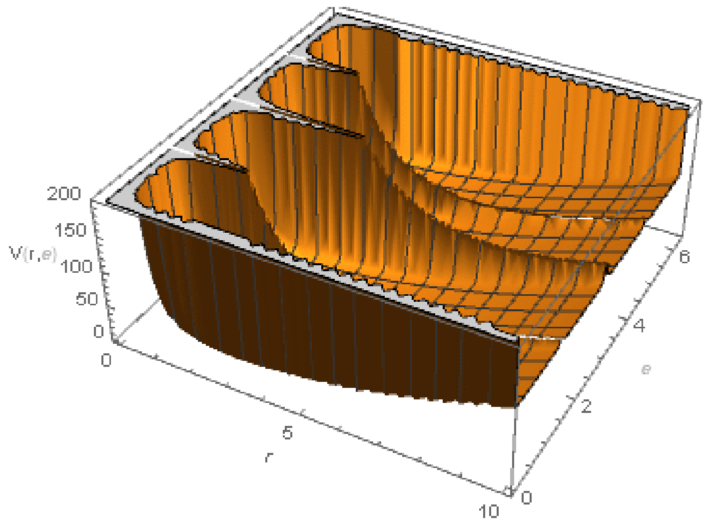

Figure 2.7: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^4\theta + \beta\cos^2\theta + \gamma)\sin^{-2}\theta\cos^{-2}\theta$ in terms of r and θ

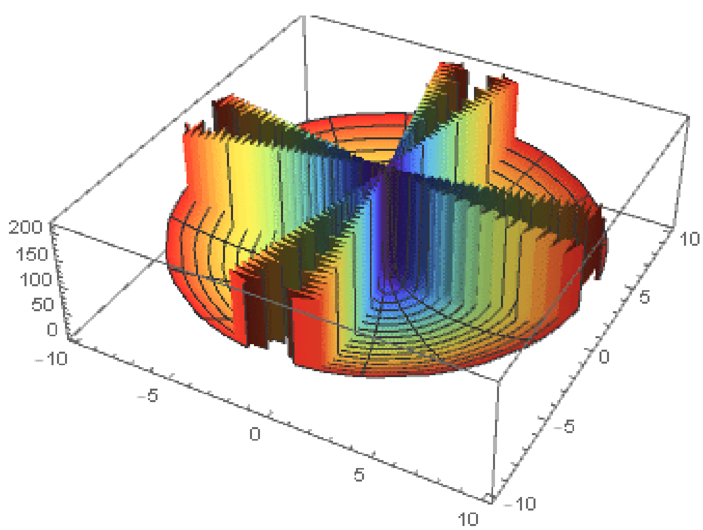

Figure 2.8: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^4\theta + \beta\cos^2\theta + \gamma)\sin^{-2}\theta\cos^{-2}\theta$ in terms of r and θ in cylindrical coordinates system

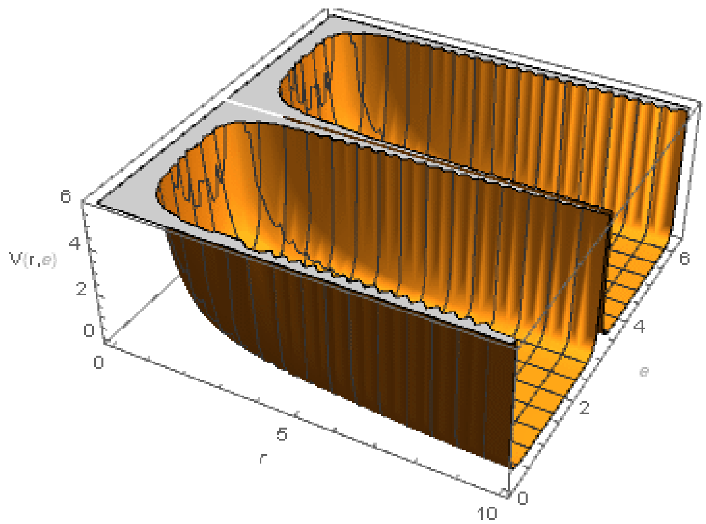

Figure 2.9: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cot^2\theta + \beta\cot\theta + \gamma)$ in terms of r and θ

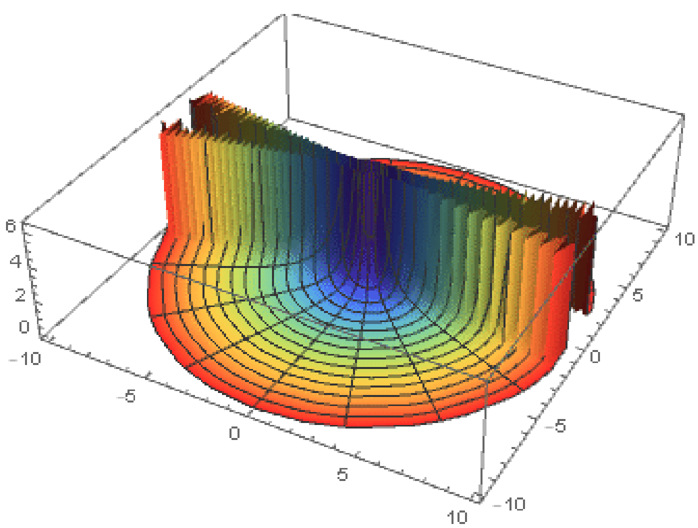

Figure 2.10: $V(r,\theta) = -\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cot^2\theta + \beta\cot\theta + \gamma)$ in terms of r and θ in cylindrical coordinates system

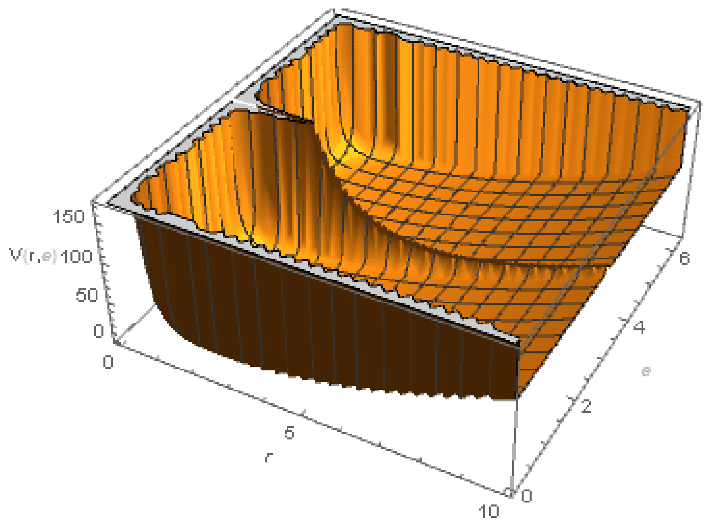

Figure 2.11: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha \cot^2\theta + \beta\cot\theta + \gamma)$ in terms of r and θ

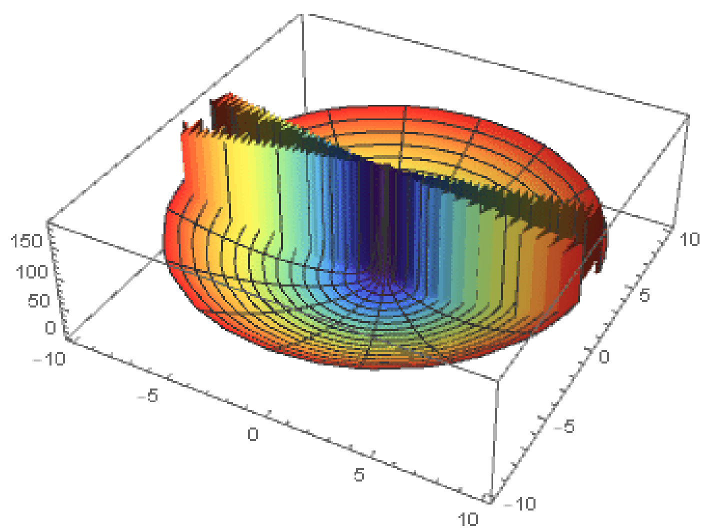

Figure 2.12: $V(r,\theta) = kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha \cot^2\theta + \beta\cot\theta + \gamma)$ in terms of r and θ in cylindrical coordinats system

2.2 Non-Relativistic Studies of 3D Non-Central Potentials

2.2.1 3D Schrödinger Equation

The Schrödinger equation is written as

$$\left[\frac{-\hbar^2}{2\mu}\Delta + V(r,\theta)\right]\psi = E\psi \tag{2.1}$$

When we substitute the potential by its expression the Schrödinger equation of our system is

$$\left[\frac{-\hbar^2}{2\mu}\Delta + \mu\left(V(r) + \frac{f(\theta)}{r^2}\right)\right]\psi = E\psi \tag{2.2}$$

To separate the variables it is better the using of the spherical coordinates (r,θ,φ) then the Schrödinger equation is written as

$$\left[\frac{-\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + \mu V(r) + \frac{\mu f(\theta)}{r^2}\right]\psi = E\psi \tag{2.3}$$

We put the equation in the more convenient following form:

$$\left[\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{2\mu^2}{\hbar^2}V(r)\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} - \frac{2\mu^2}{\hbar^2}\,f(\theta)\right)\right]\psi = -\frac{2\mu E}{\hbar^2}\psi \tag{2.4}$$

The variables can be separated when the wave function is written as :$\psi = \exp(im\varphi)\,R(r)\Theta(\theta)$,so we have to calculate the derivatives of the wave function in new expression

The first derivative of ψ with respect to r is

$$\frac{\partial\psi}{\partial r} = \frac{\partial R(r)}{\partial r}\exp(im\varphi)\,\Theta(\theta) \tag{2.5}$$

The second derivative of ψ with respect to r is

$$\frac{\partial^2\psi}{\partial r^2} = \frac{\partial^2 R(r)}{\partial r^2}\exp(im\varphi)\,\Theta(\theta) \tag{2.6}$$

The first derivative of ψ with respect to θ is

$$\frac{\partial\psi}{\partial\theta} = \frac{\partial\Theta(\theta)}{\partial\theta}\exp(im\varphi)\,R(r) \tag{2.7}$$

The second derivative of ψ with respect to θ is

$$\frac{\partial^2\psi}{\partial\theta^2} = \frac{\partial^2\Theta(\theta)}{\partial r^2}\exp(im\varphi)\,R(r) \tag{2.8}$$

The first derivative of ψ with respect to φ is

$$\frac{\partial\psi}{\partial\varphi} = im\exp\left(im\varphi\right)R(r)\Theta(\theta) \tag{2.9}$$

The first second of ψ with respect to φ is

$$\frac{\partial^2\psi}{\partial\theta^2} = -m^2\exp\left(im\varphi\right)R(r)\Theta(\theta) \tag{2.10}$$

We substitute the equations 2.5 to 2.10 in the Schrödinger equation 2.4,we find

$$\begin{gathered}\frac{\partial^2 R(r)}{\partial r^2}\exp\left(im\varphi\right)\Theta(\theta) + \frac{1}{r}\frac{\partial R(r)}{\partial r}\exp\left(im\varphi\right)\Theta(\theta) + \left(\frac{2\mu E}{\hbar^2} - \frac{2\mu^2}{\hbar^2}V(r)\right)\exp\left(im\varphi\right)R(r)\Theta(\theta) + \\ \left(\frac{1}{r^2}\frac{\partial^2\Theta(\theta)}{\partial r^2}\exp\left(im\varphi\right)R(r) + \cot\theta\frac{\partial\Theta(\theta)}{\partial\theta}\exp\left(im\varphi\right)R(r) - \right. \\ \left.\frac{m^2}{\sin^2\theta}\exp\left(im\varphi\right)R(r)\Theta(\theta) - \frac{2\mu^2}{\hbar^2}\ f(\theta)\exp\left(im\varphi\right)R(r)\Theta(\theta)\right) = 0\end{gathered} \tag{2.11}$$

We divide by $\exp\left(im\varphi\right)$

$$\begin{gathered}\left[\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \left(\frac{2\mu E}{\hbar^2} - \frac{2\mu^2}{\hbar^2}V(r)\right)R(r)\right]\Theta(\theta) + \\ \frac{1}{r^2}\left[\frac{\partial^2\Theta(\theta)}{\partial r^2} + \cot\theta\frac{\partial\Theta(\theta)}{\partial\theta} - \frac{m^2}{\sin^2\theta}\Theta(\theta) - \frac{2\mu^2}{\hbar^2}\ f(\theta)\Theta(\theta)\right]R(r) = 0\end{gathered} \tag{2.12}$$

We divide by $R(r)\Theta(\theta)$,then we find

$$\begin{gathered}\frac{1}{R(r)\Theta(\theta)}\left[\frac{\partial^2 R(r)}{\partial r^2} + \frac{2}{r}\frac{\partial R(r)}{\partial r} + \frac{2\mu}{\hbar^2}\left(E - \mu V(r)\right)R(r)\right]\Theta(\theta) + \\ \frac{1}{R(r)\Theta(\theta)}\frac{1}{r^2}\left[\frac{\partial^2\Theta(\theta)}{\partial r^2} + \cot\theta\frac{\partial\Theta(\theta)}{\partial\theta} - \frac{m^2}{\sin^2\theta}\Theta(\theta) - \frac{2\mu}{\hbar^2}\ f(\theta)\Theta(\theta)\right]R(r) = 0\end{gathered} \tag{2.13}$$

And multiplying the equation by r^2we get

$$\begin{gathered}\frac{1}{R(r)}\left[r^2\frac{\partial^2 R(r)}{\partial r^2} + 2r\frac{\partial R(r)}{\partial r} + r^2\frac{2\mu}{\hbar^2}\left(E - \mu V(r)\right)R(r)\right] = \\ -\frac{1}{\Theta(\theta)}\left[\frac{\partial^2\Theta(\theta)}{\partial r^2} + \cot\theta\frac{\partial\Theta(\theta)}{\partial\theta} - \frac{m^2}{\sin^2\theta}\Theta(\theta) - \frac{2\mu^2}{\hbar^2}\ f(\theta)\Theta(\theta)\right]\end{gathered} \tag{2.14}$$

When we put the right of equation 2.14 equal to E_θ we find two equation as

$$\frac{1}{\Theta(\theta)}\left[\frac{\partial^2\Theta(\theta)}{\partial r^2} + \cot\theta\frac{\partial\Theta(\theta)}{\partial\theta} - \frac{m^2}{\sin^2\theta}\Theta(\theta) - \frac{2\mu}{\hbar^2}\ f(\theta)\Theta(\theta)\right] = E_\theta$$

$$\frac{1}{R(r)}\left[r^2\frac{\partial^2 R(r)}{\partial r^2} + 2r\frac{\partial R(r)}{\partial r} + r^2\frac{2\mu}{\hbar^2}\left(E - \mu V(r)\right)R(r)\right] = -E_\theta$$

So this give as two equations the radial equation and the angular one

$$\frac{d^2\Theta(\theta)}{dr^2} + \cot\theta\frac{d\Theta(\theta)}{d\theta} - \frac{m^2}{\sin^2\theta}\Theta(\theta) - \frac{2\mu^2}{\hbar^2}\, f(\theta)\Theta(\theta) - E_\theta\Theta(\theta) = 0 \tag{2.15}$$

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + r^2\frac{2\mu}{\hbar^2}\left(E - \mu V(r)\right)R(r) + E_\theta R(r) = 0 \tag{2.16}$$

We replaced the partial derivative ∂ with the total derivative d because the functions $R(r)$ and $\Theta(\theta)$ have single variable

We have to solve the angular equation 2.15 to find the constants E_θ and then we use these angular eigenvalues to solve the radial equation 2.16, this will give us the energies E of the system and also the wave function $\psi\,(r,\theta)$.

2.2.2 Non-Relativistic Energy and Wave Function (Applications)

Case1 $V_1\,(r,\theta) = \mu\left[-\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cos^2\theta + \beta\cos\theta + \gamma\right)\sin^{-2}\theta\right]$

Solution of Angular Equation For this case the angular equation 2.15 becomes

$$\frac{d^2\Theta(\theta)}{d\theta^2} + \cot\theta\frac{d\Theta(\theta)}{d\theta} - \frac{m^2}{\sin^2\theta}\Theta(\theta) - \left(\alpha\cos^2\theta + \beta\cos\theta + \gamma\right)\sin^{-2}\theta\Theta(\theta) - E_\theta\Theta(\theta) = 0 \tag{2.17}$$

We make the following substitutions

$$\upsilon = \cos^2\left(\frac{\theta}{2}\right) \tag{2.18}$$

And

$$\Theta = \upsilon^\rho\left(1-\upsilon\right)^\sigma T \tag{2.19}$$

So we have to compute all parts of the equation by the new variable

$$\upsilon = \cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1+\cos\theta\right) \Longrightarrow \cos\theta = 2\upsilon - 1 \Longrightarrow \cos^2\theta = \left(2\upsilon-1\right)^2 \tag{2.20}$$

And

$$\sin^2\theta = 1 - \left(2\upsilon-1\right)^2 = 4\left(1-\upsilon\right)\left(\upsilon\right) \tag{2.21}$$

From the equations above

$$\cot\theta = \frac{2\upsilon-1}{2\sqrt{\upsilon\left(1-\upsilon\right)}} \tag{2.22}$$

The first derivative of Θ with respect to θ in term of new variable υ is

$$\frac{d\Theta}{d\theta} = -\left[\sqrt{\upsilon\left(1-\upsilon\right)}\right]\frac{d\Theta}{d\upsilon} \tag{2.23}$$

The second derivative of Θ with respect to θ in term of new variable υ is

$$\frac{d^2\Theta}{d\theta^2} = \left[\frac{1}{2}-\upsilon\right]\frac{d\Theta}{d\upsilon}+\upsilon\left(1-\upsilon\right)\frac{d^2\Theta}{d\upsilon^2} \tag{2.24}$$

The first derivative $\frac{d\Theta}{d\upsilon}$ in term of the new function T is

$$\frac{d\Theta}{d\upsilon} = \left(\rho\upsilon^{\rho-1}(1-\upsilon)^{\sigma}-\sigma\upsilon^{\rho}(1-\upsilon)^{\sigma-1}\right)T+\upsilon^{\rho}(1-\upsilon)^{\sigma}\frac{dT}{d\upsilon} \tag{2.25}$$

The second derivative $\frac{d^2\Theta}{d\upsilon^2}$ in term of the new function T is

$$\begin{aligned}\frac{d^2\Theta}{d\upsilon^2} &= \left[\left(\rho\left(\rho-1\right)\upsilon^{\rho-2}(1-\upsilon)^{\sigma}-2\rho\sigma\upsilon^{\rho-1}(1-\upsilon)^{\sigma-1}+\sigma\left(\sigma-1\right)\upsilon^{\rho}(1-\upsilon)^{\sigma-1}\right)+\right]T \\ &\quad +2\left(\rho\upsilon^{\rho-1}(1-\upsilon)^{\sigma}-\sigma\upsilon^{\rho}(1-\upsilon)^{\sigma-1}\right)\frac{dT}{d\upsilon}+\upsilon^{\rho}(1-\upsilon)^{\sigma}\frac{d^2T}{d\upsilon^2}\end{aligned} \tag{2.26}$$

By substituting the results 2.20 to 2.24 in equation 2.17 we find a new angular equation in terms of the variable υ

$$\begin{aligned}&\upsilon\left(1-\upsilon\right)\frac{d^2\Theta}{d\upsilon^2}+\left(\frac{1}{2}-\upsilon\right)\frac{d\Theta}{d\upsilon}-\left(\upsilon-\frac{1}{2}\right)\frac{d\Theta}{d\upsilon}- \\ &\frac{1}{4\left(1-\upsilon\right)\left(\upsilon\right)}\left(m^2-\alpha\left(2\upsilon-1\right)^2+\beta\left(2\upsilon-1\right)+\gamma\right)\Theta(\theta)-E_{\theta}\Theta(\theta)=0\end{aligned} \tag{2.27}$$

We use 2.25 and 2.26 we get

$$\begin{aligned}&\upsilon\left(1-\upsilon\right)\left[\omega^{\rho}(1-\omega)^{\sigma}\tfrac{d^2T}{d\omega^2}+2\left(\rho\omega^{\rho-1}(1-\omega)^{\sigma}-\sigma\omega^{\rho}(1-\omega)^{\sigma-1}\right)\tfrac{dT}{d\omega}+\right. \\ &\left.\left[\left(\rho\left(\rho-1\right)\omega^{\rho-2}(1-\omega)^{\sigma}-2\rho\sigma\omega^{\rho-1}(1-\omega)^{\sigma-1}+\sigma\left(\sigma-1\right)\omega^{\rho}(1-\omega)^{\sigma-1}\right)+\right]T\right]+ \\ &\left(1-2\upsilon\right)\left[\omega^{\rho}(1-\omega)^{\sigma}\frac{dT}{d\omega}+\left(\rho\omega^{\rho-1}(1-\omega)^{\sigma}-\sigma\omega^{\rho}(1-\omega)^{\sigma-1}\right)T\right]- \\ &\left[\frac{1}{4\left(1-\upsilon\right)\left(\upsilon\right)}\left(m^2-\alpha\left(2\upsilon-1\right)^2+\beta\left(2\upsilon-1\right)+\gamma\right)\right]\omega^{\rho}(1-\omega)^{\sigma}T-E_{\theta}\omega^{\rho}(1-\omega)^{\sigma}=0\end{aligned} \tag{2.28}$$

We divide by $\upsilon^{\rho}(1-\upsilon)^{\sigma}$ we find

$$\upsilon\left(1-\upsilon\right)\frac{d^2T}{d\upsilon^2}+\left[(2\rho+1)-(2\rho+2\sigma+2)\upsilon\right]\frac{dT}{d\upsilon}+$$
$$\left[\rho\upsilon^{-1}-\sigma(1-\upsilon)^{-1}-2\rho+2\sigma\upsilon(1-\upsilon)^{-1}+\rho\left(\rho-1\right)\upsilon^{-1}\left(1-\upsilon\right)-2\rho\sigma+\sigma\left(\sigma-1\right)\upsilon\right]T-$$
$$\frac{1}{4\left(1-\upsilon\right)\left(\upsilon\right)}\left(m^2-\alpha\left(2\upsilon-1\right)^2+\beta\left(2\upsilon-1\right)+\gamma\right)T-E_\theta T=0 \tag{2.29}$$

And

We get a hypergeometric equation

$$\upsilon\left(1-\upsilon\right)\frac{d^2T}{d\upsilon^2}+\left[(2\rho+1)-(2\rho+2\sigma+2)\upsilon\right]\frac{dT}{d\upsilon}-\left[E_\theta+2\rho\sigma+\sigma+2\rho^2+\rho-\alpha+\frac{\beta}{2}\right]T=0 \tag{2.30}$$

The solution is hypergeometric function :

$$T=N_\theta F(-l,l+1+(l^2+\alpha-\beta+\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2};1+(m^2+\alpha-\beta+\gamma)^{1/2};\upsilon) \tag{2.31}$$

And

$$\rho=\frac{1}{2}(l^2+\alpha-\beta+\gamma)^{1/2} \tag{2.32}$$

This require that

$$\sigma=\frac{1}{2}(l^2+\alpha+\beta+\gamma)^{1/2} \tag{2.33}$$

We find the angular wave function when we substitute the function T in the equation $\Theta\left(\upsilon\right)=\upsilon^\rho(1-\upsilon)^\sigma T$ as

$$\Theta\left(z\right)=N_\theta\upsilon^\rho(1-\upsilon)^\sigma F(-l,l+1+(l^2+\alpha-\beta+\gamma)^{1/2}+(l^2+\alpha+\beta+\gamma)^{1/2};1+(l^2+\alpha-\beta+\gamma)^{1/2};\upsilon) \tag{2.34}$$

We use $\upsilon=\cos^2\left(\frac{\theta}{2}\right)$,so

$$\Theta\left(z\right)=N_\theta\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^\sigma$$
$$F(-l,l+1+(l^2+\alpha-\beta+\gamma)^{1/2}+(l^2+\alpha+\beta+\gamma)^{1/2};1+(l^2+\alpha-\beta+\gamma)^{1/2};\cos^2\left(\frac{\theta}{2}\right)) \tag{2.35}$$

And the constant of separation is

$$E_\theta=\frac{1}{4}+\alpha-\left[l+\frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}+\frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}+\frac{1}{2}\right]^2 \tag{2.36}$$

$l = 0, 1, 2, ...$

Solution of Radial Equation This case is of the kratzer potential The radial equation 2.16 becomes

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[\frac{2\mu}{\hbar^2}E + \left(\frac{2\mu^2}{\hbar^2}H\right)\frac{1}{r} - \left(\frac{2\mu^2}{\hbar^2}D_r - E_\theta\right)\frac{1}{r^2}\right]R(r) = 0 \tag{2.37}$$

To solve this equation we use the following change

$$\rho = \sqrt{-\frac{8\mu}{\hbar^2}E}r \tag{2.38}$$

We calculate the derivatives of $R(r)$ in the radial equation in terms of the derivatives with respect to a new variable ρ

The first derivative $\frac{dR(r)}{dr}$ can be write as

$$\frac{dR(r)}{dr} = \frac{dR(r)}{d\rho}\frac{d\rho}{dr} = \sqrt{-\frac{8\mu}{\hbar^2}E}\frac{dR(r)}{d\rho} \tag{2.39}$$

The first derivative $\frac{dR(r)}{dr}$ can be write as

$$\frac{d^2R(r)}{dr^2} = -\frac{8\mu}{\hbar^2}E\frac{d^2R(r)}{d\rho^2} \tag{2.40}$$

By this expression the radial equation becomes

$$\begin{gathered} -\frac{8\mu}{\hbar^2}E\frac{d^2R(r)}{d\rho^2} + \frac{2\sqrt{-\frac{8\mu}{\hbar^2}E}}{\rho}\sqrt{-\frac{8\mu}{\hbar^2}E}\frac{dR(r)}{d\rho} + \\ \left[\frac{2\mu}{\hbar^2}E + \left(\frac{2\mu^2}{\hbar^2}H\right)\frac{\sqrt{-\frac{8\mu}{\hbar^2}E}}{\rho} - \left(\frac{2\mu^2}{\hbar^2}D_r - E_\theta\right)\left(\frac{-\frac{8\mu}{\hbar^2}E}{\rho^2}\right)\right]R(r) = 0 \end{gathered} \tag{2.41}$$

We divide by$-\frac{8\mu}{\hbar^2}E$ we find $\rho = \sqrt{-\frac{8\mu}{\hbar^2}E}r$

$$\frac{d^2R(r)}{d\rho^2} + \frac{2}{\rho}\frac{dR(r)}{d\rho} + \left[-\frac{1}{4} - \left(\sqrt{-\frac{\mu}{2\hbar^2E}}\mu H\right)\frac{1}{\rho} - \left(\frac{2\mu^2}{\hbar^2}D_r - E_\theta\right)\left(\frac{1}{\rho^2}\right)\right]R(r) = 0 \tag{2.42}$$

We put

$$\beta(\beta+1) = \frac{2\mu^2}{\hbar^2}D_r - E_\theta \tag{2.43}$$

And

$$\alpha = -\sqrt{-\frac{\mu}{2\hbar^2 E}}\mu H \tag{2.44}$$

So the radial equation become

$$\frac{d^2R(\rho)}{d\rho^2} + \frac{2}{\rho}\frac{dR(\rho)}{d\rho} - \left[\frac{1}{4} - \frac{\alpha}{\rho} + \beta\left(\beta+1\right)\left(\frac{1}{\rho^2}\right)\right]R(\rho) = 0 \tag{2.45}$$

To solve this equation we take following substitution

$$R(\rho) = \rho^{\beta}e^{-\frac{\rho}{2}}f(\rho) \tag{2.46}$$

We have to calculate the derivatives of in terms of the derivatives of a new function The first derivative is

$$\frac{dR}{d\rho} == \frac{d\left(\rho^{\beta}e^{-\frac{\rho}{2}}f(\rho)\right)}{d\rho} = \beta\rho^{\beta-1}e^{-\frac{\rho}{2}}f(\rho) - \frac{1}{2}\rho^{\beta}e^{-\frac{\rho}{2}}f(\rho) + \rho^{\beta}e^{-\frac{\rho}{2}}\frac{df(\rho)}{d\rho} \tag{2.47}$$

The second derivative is

$$\begin{aligned}\frac{d^2R}{d\rho^2} &= \rho^{\beta}e^{-\frac{\rho}{2}}\frac{d^2f(\rho)}{d\rho^2} + \left(2\beta\rho^{\beta-1}e^{-\frac{\rho}{2}} - \rho^{\beta}e^{-\frac{\rho}{2}}\right)\frac{df(\rho)}{d\rho} \\ &+ \left(\beta\left(\beta-1\right)\rho^{\beta-2}e^{-\frac{\rho}{2}} - \beta\rho^{\beta-1}e^{-\frac{\rho}{2}} + \frac{1}{4}\rho^{\beta}e^{-\frac{\rho}{2}}\right)f(\rho)\end{aligned} \tag{2.48}$$

We substitute the results of 2.47 and 2.48 in the equation 2.45 we find

$$\begin{gathered}\rho^{\beta}e^{-\frac{\rho}{2}}\frac{d^2f(\rho)}{d\rho^2} + \left(2\beta\rho^{\beta-1}e^{-\frac{\rho}{2}} - \rho^{\beta}e^{-\frac{\rho}{2}}\right)\frac{df(\rho)}{d\rho} + \\ \left(\beta\left(\beta-1\right)\rho r^{\beta-2}e^{-\frac{\rho}{2}} - \beta\rho^{\beta-1}e^{-\frac{\rho}{2}} + \frac{1}{4}\rho^{\beta}e^{-\frac{\rho}{2}}\right)f(\rho) + \\ \frac{2}{\rho}\left(\beta\rho^{\beta-1}e^{-\frac{\rho}{2}}f(\rho) - \frac{1}{2}\rho^{\beta}e^{-\frac{\rho}{2}}f(\rho) + \rho^{\beta}e^{-\frac{\rho}{2}}\frac{df(\rho)}{d\rho}\right) - \\ \left[\frac{1}{4} - \frac{\alpha}{\rho} + \beta\left(\beta+1\right)\left(\frac{1}{\rho^2}\right)\right]\rho^{\beta}e^{-\frac{\rho}{2}}f(\rho) = 0\end{gathered} \tag{2.49}$$

We dived by $\rho^{\beta-1}e^{-\frac{\rho}{2}}$ we get

$$\begin{gathered}\rho\frac{d^2f(\rho)}{d\rho^2} + \left(2\beta-\rho\right)\frac{df(\rho)}{d\rho} + \left(\left(\beta-1\right)\beta\rho^{-1} - \beta + \frac{1}{4}\rho\right)f(\rho) + \\ \frac{2}{\rho}\left(\beta f(\rho) - \frac{1}{2}\rho f(\rho) + \rho\frac{df(\rho)}{d\rho}\right) - \left[\frac{1}{4} - \frac{\alpha}{\rho} + \quad \beta\left(\beta+1\right)\left(\frac{1}{\rho^2}\right)\right]\rho f(\rho) = 0\end{gathered} \tag{2.50}$$

After some simplification we have

$$\rho\frac{d^2f(\rho)}{d\rho^2}+(2\beta+2-\rho)\frac{df(\rho)}{d\rho}+(\alpha-\beta-1)f(\rho) \tag{2.51}$$

The last equation is well-known associated Laguerre differential equation,and the solution here is just the confluent hypergeometric function:

$$f(\rho)=N_1F_1(\alpha-\beta-1,2\beta+2,\rho) \tag{2.52}$$

From the asymptotic behavior of the confluent series ($r\to\infty\Longrightarrow {}_1F_1=0$) which lead to $\psi\to 0$ when $r\to\infty$ we find the general condition of quantization :

$$\alpha-\beta-1=n_r, n_r=1,2,3,... \tag{2.53}$$

Then

$$\alpha=n_r+\beta+1 \tag{2.54}$$

We substitute by $\alpha=-\sqrt{-\frac{\mu}{2\hbar E}}\mu H$,$\rho=\sqrt{-\frac{8\mu}{\hbar^2}E}r$ and $\beta=\alpha-n_r-1$

$$f(r)=N_r\ {}_1F_1\left(n_r,2\beta+2,\sqrt{-\frac{8\mu}{\hbar^2}E}r\right) \tag{2.55}$$

${}_1F_1\left(n_r,-2\left(\sqrt{-\frac{\mu}{2\hbar E}}\mu H+n_r\right),\sqrt{-\frac{8\mu}{\hbar^2}E}r\right)$ can be written as Laguerre polynomials of degree n_r

$$L_{n_r}^{2\beta+2}(\sqrt{-\frac{8\mu}{\hbar^2}E}r)={}_1F_1\left(n_r,-2\left(\sqrt{-\frac{\mu}{2\hbar E}}\mu H+n_r\right),\sqrt{-\frac{8\mu}{\hbar^2}E}r\right) \tag{2.56}$$

From 2.46 we have :

$$R(r)=N_r\left(\sqrt{-\frac{8\mu}{\hbar^2}E}r\right)^{\beta}\exp\left(-\frac{1}{2}\sqrt{-\frac{8\mu}{\hbar^2}E}r\right)_1F_1\left(n_r,2\beta+2,\sqrt{-\frac{8\mu}{\hbar^2}E}r\right) \tag{2.57}$$

Since

$\alpha=-\sqrt{-\frac{\mu}{2\hbar E}}\mu H$,$\rho=\sqrt{-\frac{8\mu}{\hbar^2}E}r$ and $\beta=\alpha-n_r-1,\beta(\beta+1)=\frac{2\mu^2}{\hbar^2}D_r-E_\theta$

We use the relation $\alpha=-\sqrt{-\frac{\mu}{2\hbar^2 E}}\mu H$,$\rho=\sqrt{-\frac{8\mu}{\hbar^2}E}r$,$\beta=\alpha-n_r-1$

and $\beta(\beta+1)=\frac{2\mu^2}{\hbar^2}D_r+E_\theta$ to obtain the energy of our system

$$\alpha=-\sqrt{-\frac{\mu}{2\hbar^2 E}}\mu H\Longrightarrow\alpha^2=-\frac{\mu^3}{2\hbar E}H^2\Longrightarrow E=-\frac{\mu^3}{2\hbar\alpha^2}H^2 \tag{2.58}$$

We substitute $\alpha=n_r+\beta+1$ so the energy is

$$E = -\frac{\mu^3}{2\hbar^2 (n_r + \beta + 1)^2} H^2 \tag{2.59}$$

We have to calculate β from the following equation

$$\beta(\beta+1) = \frac{2\mu^2}{\hbar^2} D_r - E_\theta \Longrightarrow \beta^2 + \beta - \left(\frac{2\mu^2}{\hbar^2} D_r - E_\theta\right) = 0 \tag{2.60}$$

This equation have to solution

$$\beta_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\left(\frac{2\mu^2}{\hbar^2} D_r - E_\theta\right)} \tag{2.61}$$

And

$$\beta_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\left(\frac{2\mu^2}{\hbar^2} D_r - E_\theta\right)} \tag{2.62}$$

The acceptable solution is the first β_1 ,we use it in the expression of the energy 2.59 we find

$$E_{n_r} = -\frac{\mu^3 H^2}{2\hbar^2}\left(n_r + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu^2}{\hbar^2} D_r - E_\theta}\right)^{-2} \tag{2.63}$$

And the radial part of wave function is

$$R(r) = N_r \left(\sqrt{-\frac{8\mu}{\hbar^2} E r}\right)^{-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\left(\frac{2\mu^2}{\hbar^2} D_r - E_\theta\right)}} \exp\left(-\frac{1}{2}\sqrt{-\frac{8\mu}{\hbar^2} E r}\right)$$
$${}_1F_1\left(n_r, 1 + \sqrt{1 + 4\left(\frac{2\mu^2}{\hbar^2} D_r - E_\theta\right)}, \sqrt{-\frac{8\mu}{\hbar^2} E r}\right) \tag{2.64}$$

Energy and Wave function of the System We substitute the constant of separation 2.36 the expression of energy 2.63 ,we find the final expression of energy as

$$E_{n_r} = -\frac{\mu^3 H^2}{2\hbar^2}\left(n_r + \frac{1}{2} + \right.$$
$$\left.\sqrt{\frac{2\mu^2}{\hbar^2} D_r - \alpha + \left[l + \frac{1}{2}(m^2 + \alpha - \beta + \gamma)^{1/2} + \frac{1}{2}(m^2 + \alpha + \beta + \gamma)^{1/2} + \frac{1}{2}\right]^2}\right)^{-2} \tag{2.65}$$

$n_r = 0, 1, 2, ...,$ $l = 0, 1, 2, ...$and $m = 0, \pm 1, \pm 2, ...$

We deduce the wave function of our system $\psi(r, \theta, \varphi) = \exp(im\varphi) R(r)\Theta(\theta)$ from the angular part 2.35 and radial part 2.64

$$\psi_1 = N\exp(im\varphi)\left(\sqrt{-\frac{8\mu}{\hbar^2}Er}\right)^{-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\left(\frac{2\mu^2}{\hbar^2}D_r-E_\theta\right)}}$$
$$\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma}\exp\left(-\frac{1}{2}\sqrt{-\frac{8\mu}{\hbar^2}Er}\right)$$
$${}_1F_1\left(-n_r, 1+\sqrt{1+4\left(\frac{2\mu^2}{\hbar^2}D_r-E_\theta\right)}, \sqrt{-\frac{8\mu}{\hbar^2}Er}\right)\times$$
$$F(-l, l+1+(m^2+\alpha-\beta+\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2}; 1+(m^2+\alpha-\beta+\gamma)^{1/2}; \cos^2\left(\frac{\theta}{2}\right)) \tag{2.66}$$

Where $\rho = \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}$,$\sigma = \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}$

For the potential $\boldsymbol{V}_2(r,\theta) = \boldsymbol{\mu}\left[-\frac{H}{r}+\frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta+\beta\cos\theta+\gamma)\sin^{-2}\theta\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_1(r,\theta)$ when we put $D_r \longrightarrow 0$ so

$$E_2 = -\frac{\mu^3H^2}{2\hbar^2}\left(n_r+\frac{1}{2}+\right.$$
$$\left.\sqrt{-\alpha+\left[l+\frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}+\frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}+\frac{1}{2}\right]^2}\right)^{-2} \tag{2.67}$$

$n_r = 0,1,2,...,$ $l = 0,1,2,...$and $m = 0,\pm1,\pm2,...$

$$\psi_2 = N\exp(im\varphi)\left(\sqrt{-\frac{8\mu}{\hbar^2}Er}\right)^{-\frac{1}{2}+\frac{1}{2}\sqrt{1-4E_\theta}}$$
$$\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma}\exp\left(-\frac{1}{2}\sqrt{-\frac{8\mu}{\hbar^2}Er}\right)$$
$${}_1F_1\left(n_r, 1+\sqrt{1-4E_\theta}, \sqrt{-\frac{8\mu}{\hbar^2}Er}\right)\times$$
$$F(-l, l+1+(m^2+\alpha-\beta+\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2}; 1+(m^2+\alpha-\beta+\gamma)^{1/2}; \cos^2\left(\frac{\theta}{2}\right)) \tag{2.68}$$

We can studied in this case the limit at $\alpha=\beta=0$ where the potential is the ring-shaped potential $\boldsymbol{V}(r,\theta) = \boldsymbol{\mu}\left[-\frac{H}{r}+\frac{D_r}{r^2}+\frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\frac{\gamma}{\sin^2\theta}\right]$,this potential has an application field in quantum chemistry as a model for the Benzene molecule

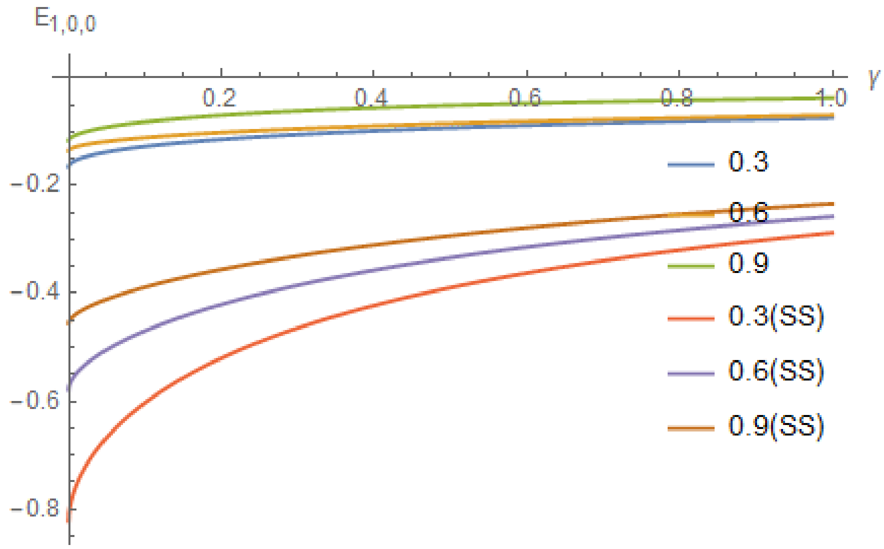

Figure 2.13: The non-relativistic energy and the non-relativistic limit energy of spin symmetry case $E_{1,0,0}$ $(n = 1, l = 0, m = 0)$ of Kratzer +ring-shaped potential in terms of γ and for $D_r = 0.3, 0.6$ *and* 0.9

$$E_{RS} = -\frac{\mu^3 H^2}{2\hbar^2}\left(n_r + \frac{1}{2} + \sqrt{\frac{2\mu^2}{\hbar^2}D_r + \left[l + (m^2+\gamma)^{1/2} + \frac{1}{2}\right]^2}\right)^{-2} \tag{2.69}$$

If we take the Colombian limit $\gamma = 0$ and compare it by the energy of hydrogen atom we find $n_r + 1 + l + m = n \Longrightarrow n_r = n - 1 - l - m$,so the energy can be written as

$$E_{RS} = -\frac{\mu^3 H^2}{2\hbar^2}\left(n - l - m - \frac{1}{2} + \sqrt{\frac{2\mu^2}{\hbar^2}D_r + \left[l + (m^2+\gamma)^{1/2} + \frac{1}{2}\right]^2}\right)^{-2} \tag{2.70}$$

$n = 1, 2, ..., \ l = 0, 1, 2, ...$and $m = 0, \pm1, \pm2, ...$

We noted that the expression under the root is always positive that means we haven't a critical values for γ and D_r

By using the Hartree units the last equation becomes

$$E_{RS} = -\frac{1}{2}\left(n - l - m - \frac{1}{2} + \sqrt{2D_r + \left[l + (m^2+\gamma)^{1/2} + \frac{1}{2}\right]^2}\right)^{-2} \tag{2.71}$$

We plotted the variation of this energy in terms of γ the parameter of the ring-shaped potential and for different values of radial momentum D_r $(Figures\ 2.13, ..., 2.15)$

From the radial equation we can plotted the effective potential ,for the ring-shaped potential was showing in $(Figures 2.16, ..., 2.19)$,we note that the state of ring shaped potential are bounded Whatever the energy level,and it is not affected by the radial momentum D_r or the parameter γ of ring-shaped potential

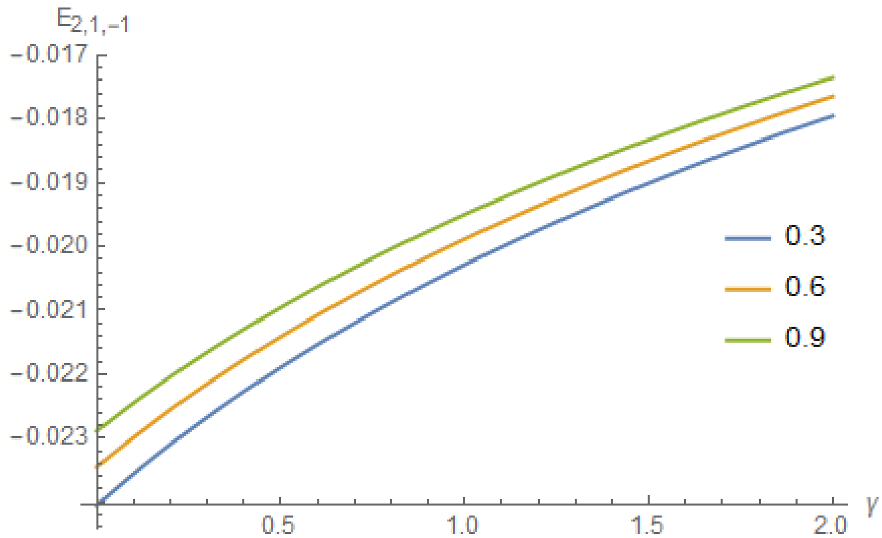

Figure 2.14: The energy $E_{2,1,-1}$ $(n = 2, l = 1, m = -1)$of Kratzer+ ring-shaped potential in terms of γ and for $D_r = 0.3, 0.6$ *and* 0.9

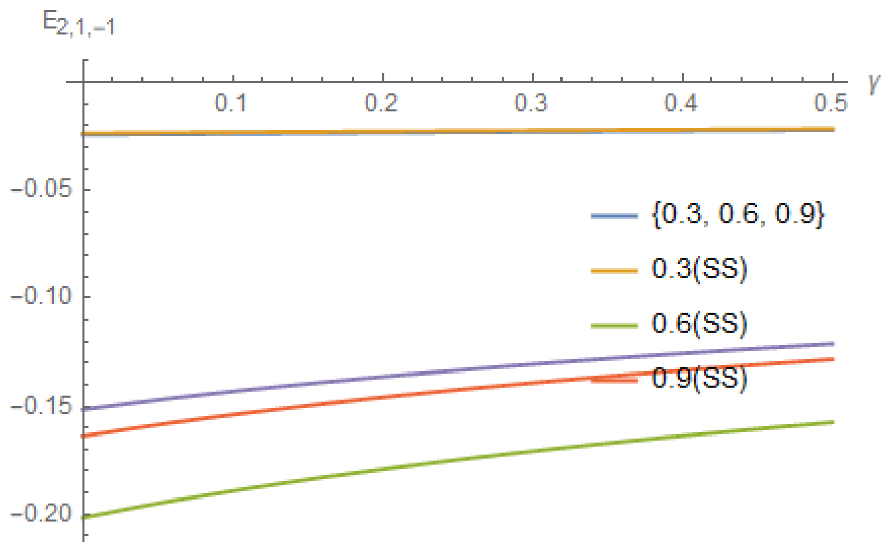

Figure 2.15: The non-relativistic energy and the non-relativistic limit energy of spin symmetry case $E_{2,1,-1}$ $(n = 2, l = 1, m = -1)$of Kratzer +ring-shaped potential in terms of γ and for $D_r = 0.3, 0.6$ *and* 0.9

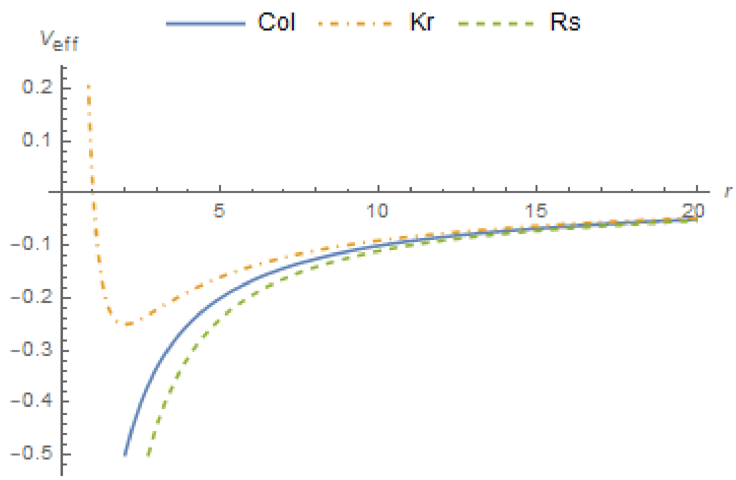

Figure 2.16: V_{eff} the effective potential of Colombian, kratzer and kratzer +ring-shaped for $l = 0, m = 0, D_r = 1, \gamma = 1$ in terms of r

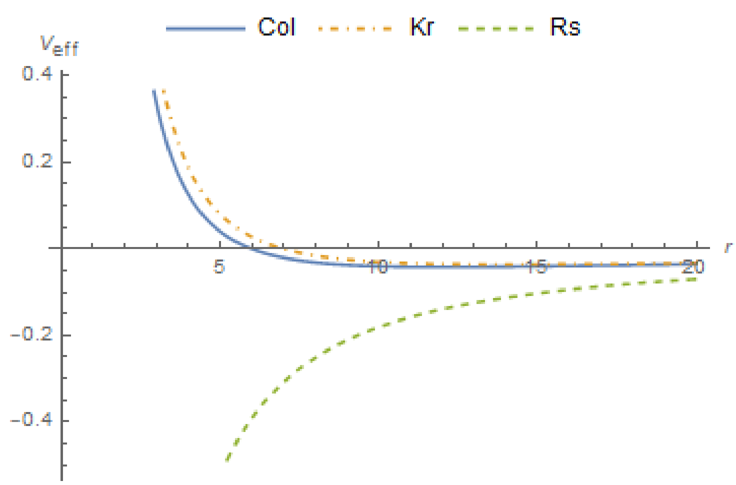

Figure 2.17: V_{eff} the effective potential of Colombian kratzer and kratzer +ring-shaped for $l = 2, m = 1, D_r = 1, \gamma = 1$ in terms of r

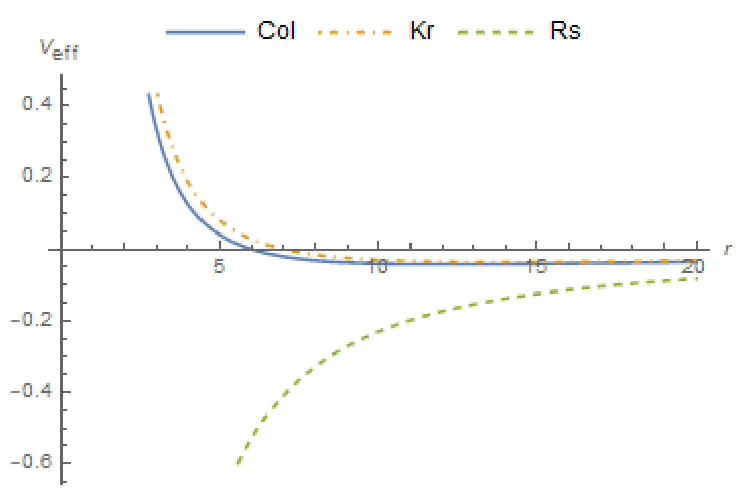

Figure 2.18: V_{eff} the effective potential of Colombian ,kratzer and kratzer +ring-shaped for $l = 2, m = 1, D_r = 1, \gamma = 3$ in terms of r

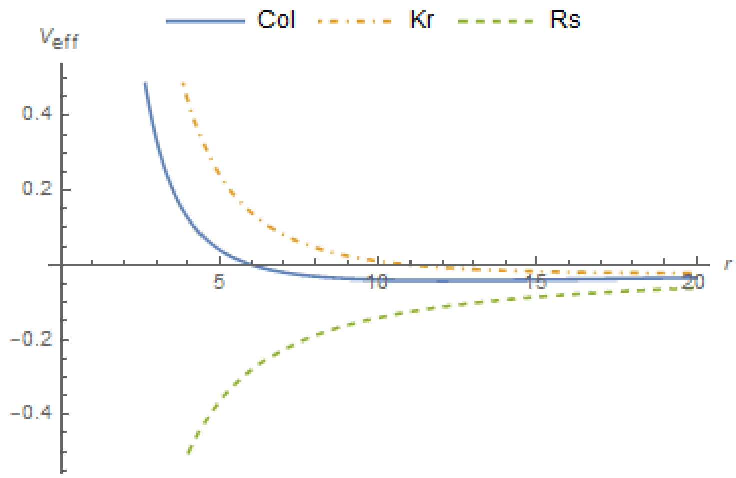

Figure 2.19: V_{eff} the effective potential of Colombian ,kratzer and kratzer +ring-shaped for $l = 2, m = 1, D_r = 3, \gamma = 1$ in terms of r

Case2 $V_3(r,\theta) = \mu\left[kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta + \beta\cos\theta + \gamma)\sin^{-2}\theta\right]$

Solution of Angular Equation The constant of separation and the angular part of wave function is the same of case1

Solution of Radial Equation This case is of pseudoharmonic oscillator potential,the radial equation 2.16 in this case becomes

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[\frac{2\mu}{\hbar^2}E - \frac{2\mu^2}{\hbar^2}kr^2 - \frac{1}{r^2}\left(\frac{2\mu^2}{\hbar^2}D_r - E_\theta\right)\right]R(r) = 0 \tag{2.72}$$

Using the dimensionless abbreviations

$$\alpha^2 = \frac{2\mu}{\hbar^2}E \tag{2.73}$$

And

$$\beta(\beta+1) = \frac{2\mu^2}{\hbar^2}D_r - E_\theta \tag{2.74}$$

So the radial equation 2.16 becomes

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[\alpha^2 - \frac{2\mu^2}{\hbar^2}kr^2 - \frac{1}{r^2}(\beta(\beta+1))\right]R(r) = 0 \tag{2.75}$$

According to the asymptotic behaviors of the radial wave functions as$r \to 0$and $r \longrightarrow \infty$,the physically acceptable solution of $R(r)$ can be expressed as

To solve this equation we make the following change

$$R(r) = r^\beta e^{-\lambda r^2} f(r) \tag{2.76}$$

Now we calculate the derivatives of $R(r)$,the first derivative is

$$\frac{dR(r)}{dr}=\left(\beta r^{\beta-1}e^{-\lambda r^2}-2\lambda r^{\beta+1}e^{-\lambda r^2}\right)f\left(r\right)+r^{\beta}e^{-\lambda r^2}\frac{df\left(r\right)}{dr} \tag{2.77}$$

The second derivative is

$$\frac{d^2R(r)}{dr^2}=r^{\beta}e^{-\lambda r^2}\frac{d^2f\left(r\right)}{dr^2}+\left(2\beta r^{\beta-1}e^{-\lambda r^2}-4\lambda r^{\beta+1}e^{-\lambda r^2}\right)\frac{df\left(r\right)}{dr}+$$
$$\left(\beta\left(\beta-1\right)r^{\beta-2}e^{-\lambda r^2}-2\lambda\beta r^{\beta}e^{-\lambda r^2}-2\lambda\left(\beta+1\right)r^{\beta}e^{-\lambda r^2}+4\lambda\beta r^{\beta+2}e^{-\lambda r^2}\right)f\left(r\right) \tag{2.78}$$

Substituting in equation 2.75 we find

$$\begin{gathered}r^{\beta}e^{-\lambda r^2}\frac{d^2f\left(r\right)}{dr^2}+\left(2\beta r^{\beta-1}e^{-\lambda r^2}-4\lambda r^{\beta+1}e^{-\lambda r^2}\right)\frac{df\left(r\right)}{dr}+\\ \left(\beta\left(\beta-1\right)r^{\beta-2}e^{-\lambda r^2}-2\lambda\beta r^{\beta}e^{-\lambda r^2}-2\lambda\left(\beta+1\right)r^{\beta}e^{-\lambda r^2}+4\lambda\beta r^{\beta+2}e^{-\lambda r^2}\right)f\left(r\right)\\ +\frac{2}{r}\left[\left(\beta r^{\beta-1}e^{-\lambda r^2}-2\lambda r^{\beta+1}e^{-\lambda r^2}\right)f\left(r\right)+r^{\beta}e^{-\lambda r^2}\frac{df\left(r\right)}{dr}\right]+\\ \left[\alpha^2-\frac{2\mu^2}{\hbar^2}kr^2-\frac{1}{r^2}\left(\beta\left(\beta+1\right)\right)\right]r^{\beta}e^{-\lambda r^2}f\left(r\right)=0\end{gathered} \tag{2.79}$$

We divide by $r^{\beta}e^{-\lambda r^2}$we find

$$\frac{d^2f\left(r\right)}{dr^2}+\left(\left(2\beta+2\right)r^{-1}-4\lambda r\right)\frac{df\left(r\right)}{dr}+\left(-\frac{2\mu^2}{\hbar^2}kr^2-2\lambda\left(2\beta+3\right)+\alpha^2+4\lambda\beta r^2\right)f\left(r\right)=0 \tag{2.80}$$

We put

$$4\lambda^2=\frac{2\mu^2}{\hbar^2}k\Longrightarrow\lambda=\sqrt{\frac{\mu^2k}{2\hbar^2}} \tag{2.81}$$

The equation 2.75 becomes

$$\frac{d^2f\left(r\right)}{dr^2}+\left(\left(2\beta+2\right)r^{-1}-4\sqrt{\frac{\mu k}{2\hbar^2}}r\right)\frac{df\left(r\right)}{dr}-\left(2\lambda\left(2\beta+3\right)\sqrt{\frac{2\mu k}{\hbar^2}}-\alpha^2\right)f\left(r\right)=0 \tag{2.82}$$

We take

$$\rho=2\lambda r^2\Longrightarrow r=\sqrt{\frac{\rho}{2\lambda}}\Longrightarrow\frac{d\rho}{dr}=2\sqrt{2\lambda\rho} \tag{2.83}$$

We calculate the derivative of $f\left(r\right)$,the first derivative is

$$\frac{df\left(r\right)}{dr}=2\sqrt{2\lambda\rho}\frac{df\left(\rho\right)}{d\rho} \tag{2.84}$$

The second derivative is

$$\frac{d^2 f(r)}{dr^2} = \frac{d}{dr}\left(\frac{df\,(r)}{dr}\right) = 2\sqrt{2\lambda\rho}\frac{d}{d\rho}\left(2\sqrt{2\lambda\rho}\frac{df\,(\rho)}{d\rho}\right) = 8\lambda\rho\frac{d^2 f(\rho)}{d\rho^2} + 4\lambda\frac{df(\rho)}{d\rho} = \tag{2.85}$$

Then we substitute this derivatives in equation 2.82 so we get the following equation :

$$8\lambda\rho\frac{d^2 f(\rho)}{d\rho^2} + (4\lambda\,(2\beta+3) - 8\lambda\rho)\,\frac{df\,(\rho)}{d\rho} + \left(-2\lambda\,(2\beta+3) + \alpha^2\right) f\,(r) = 0 \tag{2.86}$$

By dividing the last equation by 8λ we find

$$\rho\frac{d^2 f(\rho)}{d\rho^2} + \left(\frac{(2\beta+3)}{2} - \rho\right)\frac{df\,(\rho)}{d\rho} - \left(\frac{1}{4}\,(2\beta+3) - \frac{1}{8}\sqrt{\frac{2\hbar^2}{\mu^2 k}}\alpha^2\right) f\,(\rho) = 0 \tag{2.87}$$

Equation 2.87 is the Kummer's (confluent hypergeometric) differential equation and the solution of this equation that is regular at $r = 0$ or $\rho = 0$ is the degenerate hypergeometric function or the Kummer's function :

$$f\,(\rho) = N_r\ {}_1F_1\Big(\frac{1}{4}\,(2\beta+3) - \frac{1}{8}\sqrt{\frac{2\hbar^2}{\mu^2 k}}\alpha^2, \frac{(2\beta+3)}{2}, \rho\Big)\ ;\ n_r = 0, 1, 2, ... \tag{2.88}$$

And

$$f\,(r) = N_{r1}F_1\Big(\frac{1}{4}\,(2\beta+3) - \frac{1}{8}\sqrt{\frac{2\hbar^2}{\mu^2 k}}\alpha^2, \frac{(2\beta+3)}{2}, 2\sqrt{\frac{\mu^2 k}{2\hbar^2}}r^2\Big) \tag{2.89}$$

We calculate the radial wave function from the relation $R(r) = r^\beta e^{-\lambda r^2} f\,(r)$

$$R(r) = N_r r^\beta e^{-\lambda r^2}{}_1F_1\Big(\frac{1}{4}\,(2\beta+3) - \frac{1}{8}\sqrt{\frac{2\hbar^2}{\mu^2 k}}\alpha^2, \frac{(2\beta+3)}{2}, 2\sqrt{\frac{\mu^2 k}{2\hbar^2}}r^2\Big) \tag{2.90}$$

For large values of ρ, the solution in 2.87 diverges as $exp(r^2)$, thus preventing normalization,except for

$$n_r = \frac{1}{4}\left[\frac{1}{2}\sqrt{\frac{2\hbar^2}{\mu^2 k}}\alpha^2 - 2\beta - 3\right], n_r = 0, 1, 2, ... \tag{2.91}$$

From the relation $\alpha^2 = \frac{2\mu}{\hbar^2}E$, and 2.88 we have

$$E = \frac{\hbar^2}{2\mu}\left[2\sqrt{\frac{\mu^2 k}{2\hbar^2}}\,(4n_r + 2\beta + 3)\right] \tag{2.92}$$

We solve the equation $\beta\,(\beta+1) = \frac{2\mu^2}{\hbar^2}D_r - E_\theta$ to find β and use it to find the energy,so we have to solution

$$\beta_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\left(\frac{2\mu^2}{\hbar^2}D_r - E_\theta\right)} \tag{2.93}$$

And

$$\beta_1 = -\frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\left(\frac{2\mu^2}{\hbar^2}D_r - E_\theta\right)} \tag{2.94}$$

The acceptable solution is the first one β_1 ,so the energy of the system becomes

$$E = \hbar\sqrt{2k}\left(2n_r + 1 + \sqrt{\frac{1}{4} + \frac{2\mu^2}{\hbar^2}D_r - E_\theta}\right) \tag{2.95}$$

And the wave function becomes

$$\begin{aligned} R(r) = N_r r^{-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\left(\frac{2\mu^2}{\hbar^2}D_r - E_\theta\right)}} e^{-\sqrt{\frac{\mu^2 k}{2\hbar^2}}r^2} {}_1F_1(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu^2}{\hbar^2}D_r - E_\theta} - \\ 4\sqrt{\frac{\mu}{k}}E, 1 + \sqrt{\frac{1}{4} + \frac{2\mu^2}{\hbar^2}D_r - E_\theta}, 2\sqrt{\frac{\mu^2 k}{2\hbar^2}}r^2) \end{aligned} \tag{2.96}$$

We substitute the constant of separation 2.36 the expression of energy 2.95 ,we find the final expression of energy as

$$E = \hbar\sqrt{2k}\left(2n_r + 1 + \sqrt{\frac{2\mu^2}{\hbar^2}D_r - \alpha + \left[l + \frac{1}{2}(m^2 + \alpha - \beta + \gamma)^{1/2} + \frac{1}{2}(m^2 + \alpha + \beta + \gamma)^{1/2} + \frac{1}{2}\right]^2}\right) \tag{2.97}$$

$n_r = 0, 1, 2, ...,$ $l = 0, 1, 2, ...$and $m = 0, \pm1, \pm2, ...$

We deduce the wave function of our system $\psi(r, \theta, \varphi) = \exp(im\varphi) R(r)\Theta(\theta)$ from the angular part 2.35 and radial part 2.96

$$\begin{aligned} \psi_3 = N \exp(im\varphi) \cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1 - \cos^2\left(\frac{\theta}{2}\right)\right)^\sigma r^{-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\left(\frac{2\mu^2}{\hbar^2}D_r - E_\theta\right)}} e^{-\sqrt{\frac{\mu^2 k}{2\hbar^2}}r^2} \times \\ {}_1F_1(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu^2}{\hbar^2}D_r - E_\theta} - 4\sqrt{\frac{\mu}{k}}E, 1 + \sqrt{\frac{1}{4} + \frac{2\mu^2}{\hbar^2}D_r - E_\theta}, 2\sqrt{\frac{\mu^2 k}{2\hbar^2}}r^2) \\ F\left(-l, l + 1 + (m^2 + \alpha - \beta + \gamma)^{1/2} + (m^2 + \alpha + \beta + \gamma)^{1/2};\right. \\ \left. 1 + (m^2 + \alpha - \beta + \gamma)^{1/2}; \cos^2\left(\frac{\theta}{2}\right)\right) \end{aligned} \tag{2.98}$$

Where $\rho = \frac{1}{2}(m^2 + \alpha - \beta + \gamma)^{1/2}$and $\sigma = \frac{1}{2}(m^2 + \alpha + \beta + \gamma)^{1/2}$

Furthermore we can studied in this case the limit at $\alpha = \beta = 0$,and $k = \frac{1}{2}\omega$ where the po-

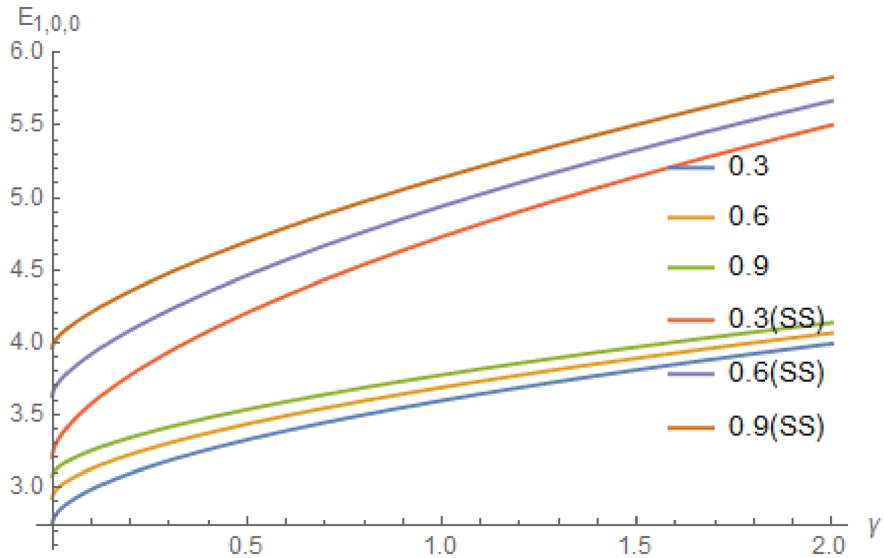

Figure 2.20: The non-relativistic energy and the non-relativistic limit energy of spin symmetry case for PHO+ring shaped potential for $n=1, l=0, m=0,\ D_r=0.3, 0.6, 0.9$ and $\omega=1$in terms of γ

tential is the pseudoharmonic ring-shaped potential $V(r,\theta)=\mu\left[\frac{1}{2}\omega r^2+\frac{D_r}{r^2}+\frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\frac{\gamma}{\sin^2\theta}\right]$,this potential has an application field in quantum chemistry as a model for the Benzene molecule the energy of this system is

$$E_{PHO+RS}=\hbar\omega\left(2n_r+1+\sqrt{\frac{2\mu^2}{\hbar^2}D_r+\left[l+(m^2+\gamma)^{1/2}+\frac{1}{2}\right]^2}\right) \tag{2.99}$$

The limit of harmonic oscillator is deduced where $D_r=\gamma=0$

$$E=\hbar\omega\left(2n_r+l+m+\frac{3}{2}\right) \tag{2.100}$$

Comparing to the energy of 3D harmonic oscillator $E=\hbar\omega\left(n+\frac{3}{2}\right)$ we find $2n_r+l+m=n\Longrightarrow 2n_r=n-l-m$,so the energy becomes

$$E_{PHO+RS}=\hbar\omega\left(n-l-m+1+\sqrt{\frac{2\mu^2}{\hbar^2}D_r+\left[l+(m^2+\gamma)^{1/2}+\frac{1}{2}\right]^2}\right) \tag{2.101}$$

In Hartree system of units the energy was written as

$$E_{PHO+RS}=\omega\left(n-l-m+1+\sqrt{2D_r+\left[l+(m^2+\gamma)^{1/2}+\frac{1}{2}\right]^2}\right) \tag{2.102}$$

$n=0,1,2,...,\ l=0,1,2,...$and $m=0,\pm1,\pm2,...$

The graphs of the energy are showed in ($Figures$ 2.20 and 2.21)

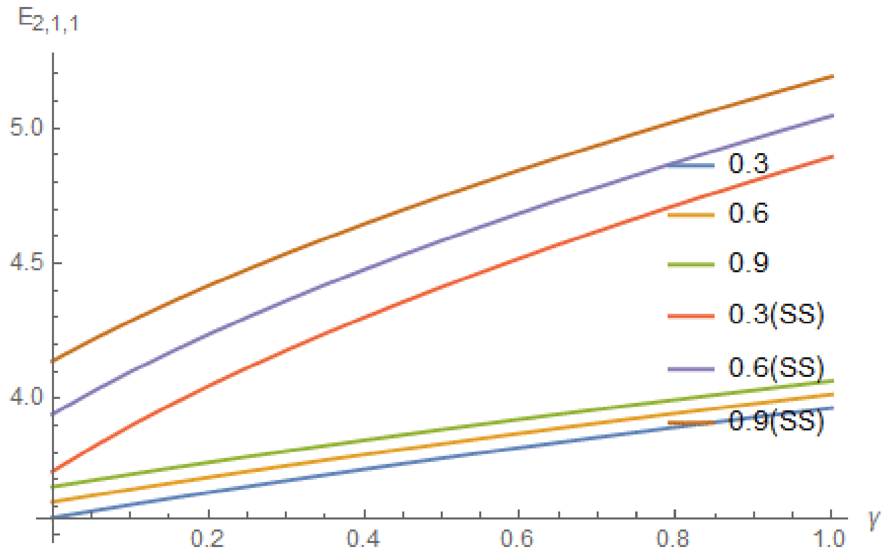

Figure 2.21: The non-relativistic energy and the non-relativistic limit energy of spin symmetry case for PHO+ring shaped potential for $n = 2, l = 1, m = 1,\ D_r = (0.3, 0.6\ and\ 0.9)$and $\omega = 1$ in terms of γ

From the graphs of effective potential $(Figures\ 2.22, ..., 2.24)$ we note that the state of ring shaped potential are bounded

For the potential $\boldsymbol{V}_4(r,\theta) = \boldsymbol{\mu}\left[kr^2 + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta + \beta\cos\theta + \gamma)\sin^{-2}\theta\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_3(r,\theta)$ when we put $D_r \longrightarrow 0$ so

$$E_4 = \hbar\sqrt{2k}\left(2n_r + 1 + \sqrt{-\alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2}\right) \tag{2.103}$$

$n_r = 0, 1, 2, ...,\ l = 0, 1, 2, ...$and $m = 0, \pm1, \pm2, ...$

The angular wave function is

$$\begin{aligned} \psi_4 = N\exp(im\varphi)\, r^{\beta} e^{-\lambda r^2}\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma} r^{-\frac{1}{2}+\frac{1}{2}\sqrt{1-4E_\theta}}\, e^{-\sqrt{\frac{\mu^2 k}{2\hbar^2}}r^2} \times \\ {}_1F_1\left(\frac{1}{2}+\sqrt{\frac{1}{4}-E_\theta} - 4\sqrt{\frac{\mu}{k}}E, 1+\sqrt{\frac{1}{4}-E_\theta}, 2\sqrt{\frac{\mu^2 k}{2\hbar^2}}r^2\right) \\ F(-l, l+1+(m^2+\alpha-\beta+\gamma)^{1/2} + (m^2+\alpha+\beta+\gamma)^{1/2}; \\ 1+(m^2+\alpha-\beta+\gamma)^{1/2}; \cos^2\left(\frac{\theta}{2}\right)) \end{aligned} \tag{2.104}$$

When $\rho = \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}$and $\sigma = \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}$

For the rest of the studied potentials,the constant of separation is obtained by the same way as a solution of hypergeometric equation and the angular part of wave function obtained as a hypergeometric function ,regarding the energy expression and the radial part of the wave

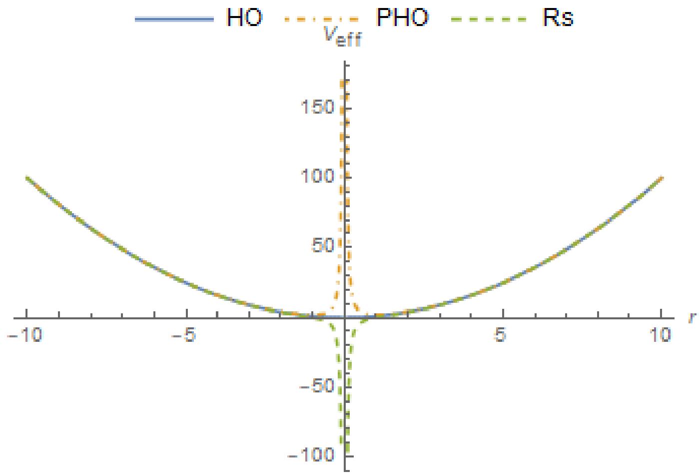

Figure 2.22: V_{eff} the effective potential of HO, PHO and PHO +ring-shaped for $l = 0, m = 0, D_r = 1, \gamma = 1$ in terms of r

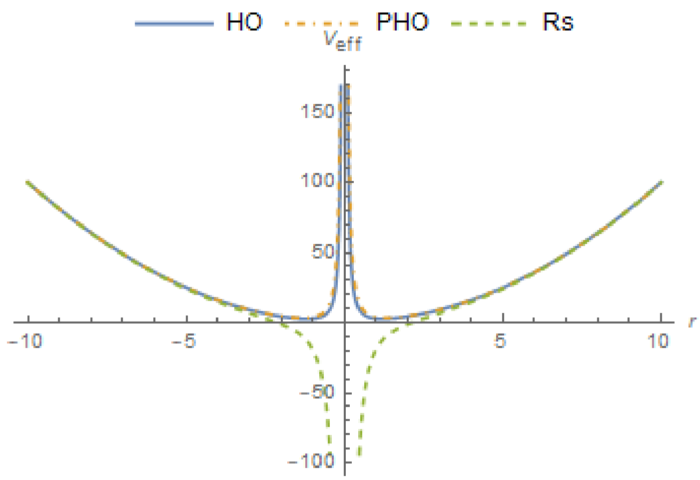

Figure 2.23: V_{eff} the effective potential of HO, PHO and PHO +ring-shaped for $l = 1, m = 1, D_r = 1, \gamma = 10$ in terms of r

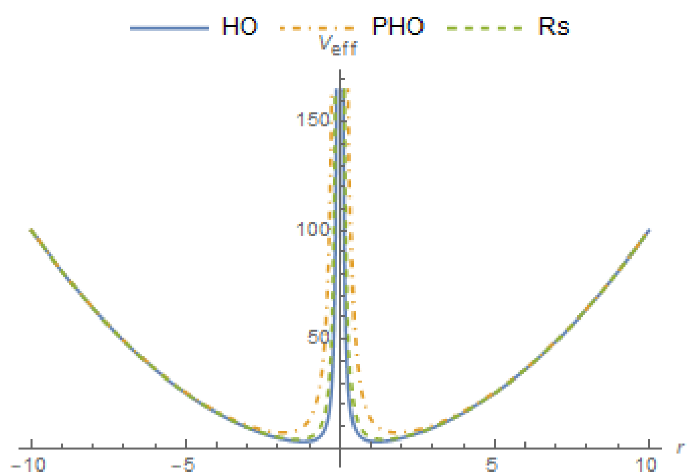

Figure 2.24: V_{eff} the effective potential of HO, PHO and PHO +ring-shaped for $l = 1, m = 1, D_r = 10, \gamma = 1$ in terms of r

$f(\theta)$	E_θ
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\dfrac{(\alpha\cos^2\theta+\beta\cos\theta+\gamma)}{\sin^2\theta}$	$\dfrac{1}{4}+\alpha-$ $\left[l+\dfrac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}+\dfrac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}+\dfrac{1}{2}\right]^2$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\dfrac{(\alpha\cos^4\theta+\beta\cos^2\theta+\gamma)}{\sin^2\theta\cos^2\theta}$	$\dfrac{1}{4}+\alpha+\left[2l+1+\dfrac{1}{2}(1+\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2}\right]^2$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $(\alpha\cot^2\theta+\beta\cot\theta+\gamma)$	$\dfrac{1}{4}-\gamma+\alpha-\dfrac{\left(2l+1+2\sqrt{m^2+\alpha}\right)^4-4\beta^2}{4\left(2l+1+2\sqrt{m^2+\alpha}\right)^2}$

Table 2.1: The 3D constant of separation

function are the same results of *case* 1 and *case* 2 .the results of the study are shown in the ($Tables$ 2.1, ..., 2.6), and a detailed calculation is provided in the *Appendix* 2

2.3 Relativistic Studies of 3D Non-Central Potentials

2.3.1 Solutions of Schrödinger Type Equation

The Spin Symmetry Case

The Schrödinger type equation for the spin-symmetry case is:

$$\left[c^2p^2+2\left(E+\mu c^2\right)U(\overrightarrow{r})-\left(E^2-\mu^2c^4\right)\right]\varphi(\overrightarrow{r})=0 \tag{2.105}$$

With the potential energy:

$$U(\overrightarrow{r})=qV(r,\theta)=q\frac{\mu}{q}\left[\frac{f(\theta)}{r^2}+V(r)\right]=\mu\left[\frac{f(\theta)}{r^2}+V(r)\right] \tag{2.106}$$

$f(\theta)$	$\Theta(\theta)$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\dfrac{(\alpha\cos^2\theta+\beta\cos\theta+\gamma)}{\sin^2\theta}$	$N_\theta\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^\sigma$ $F(-l,l+1+2\rho+2\sigma;1+2\rho;\cos^2\left(\frac{\theta}{2}\right))$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\dfrac{(\alpha\cos^4\theta+\beta\cos^2\theta+\gamma)}{\sin^2\theta\cos^2\theta}$	$N_\theta\left(\cos\theta\right)^{2\rho}\theta\left(1-\cos^2\theta\right)^\sigma$ $F(-l,l+\dfrac{1}{2}+2\rho+2\sigma;\dfrac{1}{2}+2\rho;\cos^2\theta)$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $(\alpha\cot^2\theta+\beta\cot\theta+\gamma)$	$N_\theta e^{i2\rho\theta\rho}(1-e^{2i\theta})^\sigma$ $F(-l,l+\dfrac{1}{2}+2\rho+2\sigma;\dfrac{1}{2}+2\rho;e^{2i\theta})$

Table 2.2: The 3D angular parts of wave function

$f(\theta)$	ρ	σ
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\dfrac{(\alpha\cos^2\theta+\beta\cos\theta+\gamma)}{\sin^2\theta}$	$\dfrac{1}{2}(l^2+\alpha-\beta+\gamma)^{1/2}$	$\dfrac{1}{2}(l^2+\alpha+\beta+\gamma)^{1/2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\dfrac{(\alpha\cos^4\theta+\beta\cos^2\theta+\gamma)}{\sin^2\theta\cos^2\theta}$	$\dfrac{1}{4}+\dfrac{1}{4}\left(1+4\gamma\right)^{1/2}$	$\dfrac{1}{2}(l^2+\alpha+\beta+\gamma)^{1/2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $(\alpha\cot^2\theta+\beta\cot\theta+\gamma)$	$\dfrac{1}{4}+\dfrac{1}{2}(\dfrac{1}{4}-\gamma-E_\theta+i\beta+\alpha)^{1/2}$	$(l^2+\alpha)^{1/2}$

Table 2.3: The parameters of the 3D constant of separation

$V(r)$	$R(r)$	β	ρ
$-\dfrac{H}{r}+\dfrac{D_r}{r^2}$	$N_r\left(\rho\right)^{\beta}\exp\left(\rho\right)_1 F_1\left(-n_r,2\beta+2,\rho\right)$	$-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\left(\frac{2\mu^2}{\hbar^2}D_r-E_\theta\right)}$	$\sqrt{-\frac{8\mu}{\hbar^2}E}r$
$-\dfrac{H}{r}$	$N_r\left(\rho\right)^{\beta}\exp\left(-\frac{1}{2}\rho\right)_1 F_1\left(-n_r,2\beta+2,\rho\right)$	$-\frac{1}{2}+\frac{1}{2}\sqrt{1-4E_\theta}$	$\sqrt{-\frac{8\mu}{\hbar^2}E}r$
$kr^2+\dfrac{D_r}{r^2}$	$N_r\left(r\right)^{2\alpha}\exp\left(\rho\right)_1 F_1\left(-n_r,\beta+\dfrac{3}{2},\rho\right)$	$-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\left(\frac{2\mu^2}{\hbar^2}D_r-E_\theta\right)}$	$2\sqrt{\frac{\mu^2k}{2\hbar^2}}r^2$
kr^2	$N_r\left(r\right)^{2\alpha}\exp\left(\rho\right)_1 F_1\left(-n_r,\beta+\dfrac{3}{2},\rho\right)$	$-\frac{1}{2}+\frac{1}{2}\sqrt{1-4E_\theta}$	$2\sqrt{\frac{\mu^2k}{2\hbar^2}}r^2$

Table 2.4: The 3D radial part of the wave function

$V(r,\theta)$	E_{n_r}
$\mu\left(-\dfrac{H}{r}+\dfrac{D_r}{r^2}+\dfrac{f(\theta)}{r^2}\right)$	$-\frac{\mu^3H^2}{2\hbar^2}\left(n_r+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu^2}{\hbar^2}D_r-E_\theta}\right)^{-2}$
$\mu\left(-\dfrac{H}{r}+\dfrac{f(\theta)}{r^2}\right)$	$-\frac{\mu^3H^2}{2\hbar^2}\left(n_r+\frac{1}{2}+\sqrt{\frac{1}{4}-E_\theta}\right)^{-2}$
$\mu\left(kr^2+\dfrac{D_r}{r^2}+\dfrac{f(\theta)}{r^2}\right)$	$\hbar\sqrt{2k}\left[2n_r+1+\sqrt{-E_\theta+\dfrac{2\mu^2D_r}{\hbar^2}}\right]$
$\mu\left(kr^2+\dfrac{f(\theta)}{r^2}\right)$	$\hbar\sqrt{2k}\left[2n_r+1+\sqrt{-E_\theta}\right]$

Table 2.5: Expression of 3D energy

$V(r,\theta)$	$\psi\left(r,\theta,\varphi\right)$
$\mu\left(-\frac{H}{r}+\frac{D_r}{r^2}+\frac{f(\theta)}{r^2}\right)$	$N\exp\left(im\varphi\right)\left(\rho\right)^{\beta}\exp\left(-\frac{1}{2}\rho\right)_1 F_1\left(-n_r,2\beta+2,\rho\right)\times\Theta(\theta)$
$\mu\left(-\frac{H}{r}+\frac{f(\theta)}{r^2}\right)$	$N\exp\left(im\varphi\right)\left(\rho\right)^{\beta}\exp\left(-\frac{1}{2}\rho\right)_1 F_1\left(-n_r,2\beta+2,\rho\right)\times\Theta(\theta)$
$\mu\left(kr^2+\frac{D_r}{r^2}+\frac{f(\theta)}{r^2}\right)$	$N\exp\left(im\varphi\right)\exp\left(\rho\right)_1 F_1\left(-n_r,\beta+\frac{3}{2},\rho\right)\times\Theta(\theta)$
$\mu\left(kr^2+\frac{f(\theta)}{r^2}\right)$	$N\exp\left(im\varphi\right)\exp\left(\rho\right)_1 F_1\left(-n_r,\beta+\frac{3}{2},\rho\right)\times\Theta(\theta)$

Table 2.6: The expression of 3D wave function

In the spheric coordinates the equation 2.105 becomes

$$\frac{-\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)+$$
$$2\frac{(E+\mu c^2)}{c^2}\left(\mu V(r)+\frac{\mu f(\theta)}{r^2}\right)-\frac{(E^2-\mu^2c^4)}{c^2}\psi(\overrightarrow{r})=0 \qquad (2.107)$$

We use the same transformation as before $\psi(r,\theta,\varphi)=\exp\left(im\varphi\right)R(r)\Theta(\theta)$ to get two separate equations as :

$$\frac{d^2\Theta(\theta)}{dr^2}+\cot\theta\frac{d\Theta(\theta)}{d\theta}-\frac{m^2}{\sin^2\theta}\Theta(\theta)-\frac{2\mu^2}{\hbar^2}\left[\frac{2\left(E+\mu c^2\right)}{c^2}\right]f(\theta)\Theta(\theta)-E_\theta\Theta(\theta)=0 \qquad (2.108)$$

$$r^2\frac{d^2R(r)}{dr^2}+2r\frac{dR(r)}{dr}+r^2\frac{2\mu}{\hbar^2}\left(\frac{(E^2-\mu^2c^4)}{c^2}-\mu\frac{2\left(E+\mu c^2\right)}{c^2}V(r)\right)R(r)-E_\theta R(r)=0 \qquad (2.109)$$

We compare the equation above with the equations of the non-relativistic case

$$\frac{d^2\Theta(\theta)}{dr^2}+\cot\theta\frac{d\Theta(\theta)}{d\theta}-\frac{m^2}{\sin^2\theta}\Theta(\theta)-\frac{2\mu^2}{\hbar^2}f(\theta)\Theta(\theta)-E_\theta\Theta(\theta)=0 \qquad (2.110)$$

$$r^2\frac{d^2R(r)}{dr^2}+2r\frac{dR(r)}{dr}+r^2\frac{2\mu}{\hbar^2}\left(E-\mu V(r)\right)R(r)+E_\theta R(r)=0 \tag{2.111}$$

We note that the equations of relativistic case is the same of nonrelativistic case when we use the following transformation

$$E\longrightarrow\frac{\left(E^2-\mu^2c^4\right)}{c^2} \tag{2.112}$$

$$V(r)\longrightarrow\frac{2\left(E+\mu c^2\right)}{c^2}V(r) \tag{2.113}$$

$$f(\theta)\longrightarrow\frac{2\left(E+\mu c^2\right)}{c^2}\ f(\theta) \tag{2.114}$$

So the parameters(α,β,γ) of $f(\theta)$ change to $\left(\frac{2}{c^2}\left(E+\mu c^2\right)\alpha,\frac{2}{c^2}\left(E+\mu c^2\right)\beta,\frac{2}{c^2}\left(E+\mu c^2\right)\gamma\right)$

2.3.2 Relativistic Energy and Wave Function (Applications)

When we substitute by the transformations above in the expression of nonrelativistic energy we find the equation of relativistic energy

For the kratzer potential the equation of energy is

$$\frac{\left(E^2-\mu^2c^4\right)}{c^2}=-\frac{2\mu^3}{\hbar^2}\left(\frac{E+\mu c^2}{c^2}H\right)^2\left(n_r+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{4\mu^2}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r-E_\theta}\right)^{-2} \tag{2.115}$$

For the pseudoharmonic potential the equation of energy is

$$\frac{\left(E^2-\mu^2c^4\right)}{c^2}=\hbar2\sqrt{\frac{E+\mu c^2}{c^2}k}\left(2n_r+1+\sqrt{\frac{1}{4}+\frac{4\mu^2}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r-E_\theta}\right) \tag{2.116}$$

After some simplification the last equation the energy of non_central pseudoharmonic energy becomes

$$\sqrt{\left(E+\mu c^2\right)}\left(E-\mu c^2\right)=2\hbar c\sqrt{k}\left(2n_r+1+\sqrt{\frac{1}{4}+\frac{4\mu^2}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r-E_\theta}\right) \tag{2.117}$$

Concerning the wave function and the rest of the results are shown in the $(Tables\ 2.7,...,2.12)$

For the **kratzer +ring- shaped** potential the separation constant and the energy in relativistic case fulfilling the following equations

$f(\theta)$	E_θ
$case1$	$\frac{1}{4}+2\frac{E+\mu c^2}{c^2}\alpha-\left[l+\frac{1}{2}(m^2+2\frac{E+\mu c^2}{c^2}\left(\alpha-\beta+\gamma\right))^{1/2}+\frac{1}{2}(m^2+2\frac{E+\mu c^2}{c^2}\left(\alpha+\beta+\gamma\right))^{1/2}+\frac{1}{2}\right]^2$
$case2$	$\frac{1}{4}+2\frac{E+\mu c^2}{c^2}\alpha+\left[2l+1+\frac{1}{2}\left(1+2\frac{E+\mu c^2}{c^2}\gamma\right)^{1/2}+(m^2+2\frac{E+\mu c^2}{c^2}\left(\alpha+\beta+\gamma\right))^{1/2}\right]^2$
$case3$	$\frac{1}{4}+2\frac{E+\mu c^2}{c^2}\left(\alpha-\gamma\right)-\frac{\left(2l+1+2\sqrt{m^2+2\frac{E+\mu c^2}{c^2}\alpha}\right)^4-16\left(\frac{E+\mu c^2}{c^2}\beta\right)^2}{4\left(2l+1+2\sqrt{m^2+2\frac{E+\mu c^2}{c^2}\alpha}\right)^2}$

Table 2.7: The relativistic 3D constant of separation

$f(\theta)$	$\Theta(\theta)$
$\left(\frac{\hbar^2}{2\mu^2}\right)\frac{(\alpha\cos^2\theta+\beta\cos\theta+\gamma)}{\sin^2\theta}$	$N_\theta\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^\sigma F(-l,l+1+2\rho+2\sigma;1+2\rho;\cos^2\left(\frac{\theta}{2}\right))$
$\left(\frac{\hbar^2}{2\mu^2}\right)\frac{(\alpha\cos^4\theta+\beta\cos^2\theta+\gamma)}{\sin^2\theta\cos^2\theta}$	$N_\theta\left(\cos\theta\right)^{2\rho}\theta\left(1-\cos^2\theta\right)^\sigma F(-l,l+\frac{1}{2}+2\rho+2\sigma;\frac{1}{2}+2\rho;\cos^2\theta)$
$\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cot^2\theta+\beta\cot\theta+\gamma)$	$N_\theta e^{i2\rho\theta\rho}(1-e^{2i\theta})^\sigma F(-l,l+\frac{1}{2}+2\rho+2\sigma;\frac{1}{2}+2\rho;e^{2i\theta})$

Table 2.8: The relativistic 3D angular part of wave function

$f(\theta)$	ρ	σ
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\dfrac{(\alpha\cos^2\theta+\beta\cos\theta+\gamma)}{\sin^2\theta}$	$\dfrac{1}{2}(l^2+2\frac{E+\mu c^2}{c^2}(\alpha-\beta+\gamma))^{1/2}$	$\dfrac{1}{2}(l^2+2\frac{E+\mu c^2}{c^2}(\alpha+\beta+\gamma))^{1/2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $\dfrac{(\alpha\cos^4\theta+\beta\cos^2\theta+\gamma)}{\sin^2\theta\cos^2\theta}$	$\dfrac{1}{4}+\dfrac{1}{4}\left(1+8\frac{E+\mu c^2}{c^2}\gamma\right)^{1/2}$	$\dfrac{1}{2}(l^2+2\frac{E+\mu c^2}{c^2}(\alpha+\beta+\gamma))^{1/2}$
$\left(\frac{\hbar^2}{2\mu^2}\right)$ $(\alpha\cot^2\theta+\beta\cot\theta+\gamma)$	$\dfrac{1}{4}+\dfrac{1}{2}(\dfrac{1}{4}-E_\theta+2\frac{E+\mu c^2}{c^2}(\alpha+i\beta-\gamma))^{1/2}$	$(l^2+2\frac{E+\mu c^2}{c^2}\alpha)^{1/2}$

Table 2.9: The parameters of the relativistic 3D constant of separation

$V(r)$	$R(r)$	β	ρ
$-\dfrac{H}{r}+\dfrac{D_r}{r^2}$	$N_r\,(\rho)^\beta\exp(\rho)\times$ ${}_1F_1(-n_r,2\beta+2,\rho)$	$-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\left(\frac{4\mu^2}{\hbar^2}\frac{(E+\mu c^2)}{c^2}D_r-E_\theta\right)}$	$\sqrt{-\frac{8\mu}{\hbar^2}\frac{(E^2-\mu^2c^4)}{c^2}}r$
$-\dfrac{H}{r}$	$N_r\,(\rho)^\beta\exp\left(-\frac{1}{2}\rho\right)\times$ ${}_1F_1(-n_r,2\beta+2,\rho)$	$-\frac{1}{2}+\frac{1}{2}\sqrt{1-4E_\theta}$	$\sqrt{-\frac{8\mu}{\hbar^2}\frac{(E^2-\mu^2c^4)}{c^2}}r$
$kr^2+\dfrac{D_r}{r^2}$	$N_r\,(r)^{2\alpha}\exp(\rho)\times$ ${}_1F_1\left(-n_r,\beta+\dfrac{3}{2},\rho\right)$	$-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\left(\frac{4\mu^2}{\hbar^2}\frac{(E+\mu c^2)}{c^2}D_r-E_\theta\right)}$	$2\sqrt{\frac{\mu^2}{\hbar^2}\frac{(E+\mu c^2)}{c^2}}kr^2$
kr^2	$N_r\,(r)^{2\alpha}\exp(\rho)$ ${}_1F_1\left(-n_r,\beta+\dfrac{3}{2},\rho\right)$	$-\frac{1}{2}+\frac{1}{2}\sqrt{1-4E_\theta}$	$2\sqrt{\frac{\mu^2}{\hbar^2}\frac{(E+\mu c^2)}{c^2}}kr^2$

Table 2.10: The 3D radial part of the wave function

$V(r,\theta)$	$\frac{\left(E^2-\mu^2c^4\right)}{c^2}$
$\mu\left(-\frac{H}{r}+\frac{D_r}{r^2}+\frac{f(\theta)}{r^2}\right)$	$-\frac{2\mu^3}{\hbar^2}\left(\frac{E+\mu c^2}{c^2}H\right)^2\left(n_r+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{4\mu^2}{\hbar^2}\frac{(E+\mu c^2)}{c^2}D_r-E_\theta}\right)^{-2}$
$\mu\left(-\frac{H}{r}+\frac{f(\theta)}{r^2}\right)$	$-\frac{2\mu^3}{\hbar^2}\left(\frac{E+\mu c^2}{c^2}H\right)^2\left(n_r+\frac{1}{2}+\sqrt{\frac{1}{4}-E_\theta}\right)^{-2}$
$\mu\left(kr^2+\frac{D_r}{r^2}+\frac{f(\theta)}{r^2}\right)$	$\hbar\sqrt{4\frac{(E+\mu c^2)}{c^2}k}\left[2n_r+1+\sqrt{-E_\theta+\frac{4\mu^2}{\hbar^2}\frac{(E+\mu c^2)}{c^2}D_r}\right]$
$\mu\left(kr^2+\frac{f(\theta)}{r^2}\right)$	$\hbar\sqrt{4\frac{(E+\mu c^2)}{c^2}k}\left[2n_r+1+\sqrt{-E_\theta}\right]$

Table 2.11: The relativistic 3D energy

$V(r,\theta)$	$\psi(r,\theta,\varphi)$
$\mu\left(-\frac{H}{r}+\frac{D_r}{r^2}+\frac{f(\theta)}{r^2}\right)$	$N\exp(im\varphi)(\rho)^{\beta}\exp\left(-\frac{1}{2}\rho\right)_1F_1(-n_r,2\beta+2,\rho)\times\Theta(\theta)$
$\mu\left(-\frac{H}{r}+\frac{f(\theta)}{r^2}\right)$	$N\exp(im\varphi)(\rho)^{\beta}\exp\left(-\frac{1}{2}\rho\right)_1F_1(-n_r,2\beta+2,\rho)\times\Theta(\theta)$
$\mu\left(kr^2+\frac{D_r}{r^2}+\frac{f(\theta)}{r^2}\right)$	$N\exp(im\varphi)\exp(\rho)_1F_1\left(-n_r,\beta+\frac{3}{2},\rho\right)\times\Theta(\theta)$
$\mu\left(kr^2+\frac{f(\theta)}{r^2}\right)$	$N\exp(im\varphi)\exp(\rho)_1F_1\left(-n_r,\beta+\frac{3}{2},\rho\right)\times\Theta(\theta)$

Table 2.12: The expression of 3D wave function

$$E_\theta = \frac{1}{4} - \left[l + (m^2 + \frac{2\left(E+\mu c^2\right)}{c^2}\gamma)^{1/2} + \frac{1}{2} \right]^2 \tag{2.118}$$

$$\frac{\left(E^2-\mu^2c^4\right)}{c^2} = -\frac{2\mu^3}{\hbar^2}\left(\frac{E+\mu c^2}{c^2}H\right)^2 \left(n-l-m-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{4\mu^2}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r - E_\theta}\right)^{-2} \tag{2.119}$$

We substitute E_θ in the equation of energy and after some simplification we get

$$\left(E-\mu c^2\right) = -\frac{2\mu^3H^2}{\hbar^2c^2}\left(E+\mu c^2\right)\left(n-l-m-\frac{1}{2}+\right.$$
$$\left.\sqrt{\frac{4\mu^2}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r + \left[l + (m^2 + \frac{2\left(E+\mu c^2\right)}{c^2}\gamma)^{1/2} + \frac{1}{2} \right]^2}\right)^{-2} \tag{2.120}$$

We use the relation between the relativistic and non-relativistic energy $E_{n,m} = E - \mu c^2$,where we neglecting the term $E_{n,m}$ beside the factor $2\mu c^2$

$$E_\theta = \frac{1}{4} - \left[l + (m^2 + 4\mu\gamma)^{1/2} + \frac{1}{2} \right]^2 \tag{2.121}$$

$$\left(E_{n,m}\right) = -\frac{4\mu^4H^2}{\hbar^2}\left(n-l-m-\frac{1}{2}+\sqrt{\frac{8\mu^3}{\hbar^2}D_r + \left[l + (m^2 + 4\mu\gamma)^{1/2} + \frac{1}{2} \right]^2}\right)^{-2} \tag{2.122}$$

We use the Hartree units the energy becomes

$$\left(E_{n,m}\right)_{RS} = -4\left(n-l-m-\frac{1}{2}+\sqrt{8D_r + \left[l + (m^2 + 4\gamma)^{1/2} + \frac{1}{2} \right]^2}\right)^{-2}$$

We note that this expression is different comparing by the expression of equation 2.71 by the number 8 the graphs of this energy is shown in ($Figures$ 2.13, ..., 2.15)

For the **pseudoharmonic +ring- shaped** potential we substitute the constant of separation then the energy equation is the equation of relativistic energy is $E = E_{n,m} + \mu c^2$

$$\sqrt{\left(E+\mu c^2\right)}\left(E-\mu c^2\right) = \sqrt{2}c\hbar\omega\left(n-l-m+1+\sqrt{\frac{1}{4}+\frac{4\mu^2}{\hbar^2}\frac{\left(E+\mu c^2\right)}{c^2}D_r - E_\theta}\right) \tag{2.123}$$

We substitute the relativistic energy by his expression in terms of the non-relativistic energy $E_{n,m} + \mu c^2 = E$

$$\sqrt{(E_{n,m}+2\mu c^2)}\,(E_{n,m})=\sqrt{2}c\hbar\omega\left(n-l-m+1+\frac{8\mu^3}{\hbar^2}D_r+\left[l+(m^2+4\mu\gamma)^{1/2}+\frac{1}{2}\right]^2\right) \tag{2.124}$$

by neglecting the term $E_{n,m}$ beside the factor $2\mu c^2$ we find

$$E_{n,m}=\frac{\hbar}{\sqrt{\mu}}\omega\left(n-l-m+1+\sqrt{\frac{8\mu^3}{\hbar^2}D_r+\left[l+(m^2+4\mu\gamma)^{1/2}+\frac{1}{2}\right]^2}\right) \tag{2.125}$$

In the Hartree system of units the last equation becomes

$$(E_{n,m})_{PHO+RS}=\omega\left(n-l-m+1+\sqrt{8D_r+\left[l+(m^2+4\gamma)^{1/2}+\frac{1}{2}\right]^2}\right) \tag{2.126}$$

This expression is different comparing by the expression of equation 2.101 by the number 8 the graphs of this energy is shown in $(Figures\ 2.20, ..., 2.21)$

2.4 Discussion

In this chapter, we studied some non-central potentials $V(r,\theta)=\mu\left[V(r)+\frac{f(\theta)}{r^2}\right]$ for 3D quantum systems in both non-relativistic and relativistic cases. We solved the Schrödinger equation analytically and studied the relativistic spectrum for Klein-Gordon and Dirac equations in both spin and pseudo-spin symmetry We note in this chapter that in the 3D space to find a bond state of a particle moving in noncentral potential and with the presence of kratzer or pseudoharmonic potential the following condition must be fulfilled $\frac{1}{4}+\frac{2\mu^2}{\hbar^2}D_r-E_\theta\geq 0$ this give as a critical values for the parameters of the noncentral potential and this critical value influenced by the parameters of the kratzer potential when it can get bigger or smaller ,the non-central potentials remove the degeneracy occurrence of the three quantum numbers (n_r,l,m).Concerning the relativistic case we note the energy is found in the form of a second-order equation in terms of the potential parameters and in terms of numbers(n_r,l,m), and to find its expression, we must give numerical values for the potential parameters to solve this equation,we studied the ring-shaped potential as a example The previous condition is always fulfilled and all the states of energy are bound state whatever its level or its radial momentum and this states don't affected by the parameter of the ring-shaped potential

Part II

The Quantum Studies of Some Non-Central Potentials in Deformed Space (deSitter Space and Anti deSitter Space)

Chapter 3

Studies of N-C Potentials in 2D (dS and AdS) Spaces

3.1 Introduction

Historically, at a microscopic scale of high energy, several scenarios have been proposed to study the deformed quantum mechanical systems at small scales in order to absorb the infinities vitiating the standard field theories. Notably the Snyder model; which has suggested that measurements in noncommutative quantum mechanics can be governed by a generalized uncertainty principle (GUP) [103], admitting a fundamental length scale that is supposed to be in the order of the Planck length as to that proposed by Kempf [104], leads to a nonzero minimal uncertainty in the measurement of the position. This was motivated by noncommutative geometries [105], doubly special relativity [106], string theories [107] and black hole physics [108].

On the other hand, there is a great interest on studying of the curved space–time which has important astrophysical and cosmological implications in general relativity, where gravity is described as a property of the geometry of space–time. This implication was a great advance in understanding the expansion of space and the shape of the universe. Furthermore, at the atomic scale, the study of quantum mechanical problems in curved space-time can be considered as a new kind of interaction between quantum matter and gravitation in the microparticle world. In this situation, curved space–time was a great advance in understanding the nature, dealing with the structure of the space–time which is perturbed by the gravitational field. In other way, at small length scales as a doubly special relativity (DSR) theory [106], there are many arguments on that quantum gravity implies also a minimal measurable length in the order of the Planck length as in the previous case of Snyder.

For this reason, a large amount of effort has been devoted to extend the study of quantum mechanics in the flat space Snyder model to a curved space–time generalized algebra. The idea behind this extension is to take into account the modification of the standard Heisenberg algebra by adding small corrections to the canonical commutation relations such as the generalization of the uncertainty relations (GUR)[109] and extended uncertainty principle (EUP) [110] in order to incorporate the noncommutative geometries and gravity effect, respectively into quantum mechanics. Recently, at the level of relativistic and non-relativistic quantum mechanics, some problems were solved within this framework; for example, the Schrödinger equation was exactly solved with the free particle, the harmonic oscillator and the Dirac oscillator in curved Snyder model [111, 112].

Our work will be structured as follows: In section 2, we will give a review of the Snyder-de Sitter model. In section 3,will be devoted to explain the Nikiforov–Uvarov method section 4,is the crux of this work when we will solve the Schrödinger equation of non-central Kratzer potential with Snyder model ,. The exact solution will be obtain for this equation and the energy spectrum and wave functions will be deduce The last section will be left for concluding remarks.

3.1.1 Review of the Deformed Quantum Mechanics Relations

In three-dimensional space, the deformed Heisenberg algebra leading to EUP is defined by the following commutation relations [111][112]

$$[X_i, X_j] = 0 \ , \ [P_i, P_j] = i\hbar\tau\lambda\epsilon_{ijk}L_k \ , \ [X_i, P_j] = i\hbar\left(\delta_{ij} - \tau\lambda X_i X_j\right) \text{ with } \tau = -1, +1 \quad (3.1)$$

Where λ is the parameter of the deformation and it is very small because, in the context of quantum gravity, this EUP parameter is determined as a fundamental constant associated to the scale factor of the expanding universe and it is proportional to the cosmological constant $\Gamma = 3\tau\lambda = 3\tau/a^2$ where a is the deSitter radius [113]. L_k is the component of the angular momentum expressed by:

$$L_k = \epsilon_{ijk} X_i P_j \quad (3.2)$$

And satisfying the usual algebra:

$$[L_i, P_j] = i\hbar\varepsilon_{ijk}P_k \ , \ [L_i, X_j] = i\hbar\varepsilon_{ijk}X_k \ , \ [L_i, L_j] = i\hbar\varepsilon_{ijk}L_k \quad (3.3)$$

As in ordinary quantum mechanics, the commutation relation 3.3 gives rise to a Heisenberg uncertainty relation:

$$\Delta X_i \Delta P_i \geq \frac{\hbar}{2}\left(1 - \tau\lambda\left(\Delta X_i\right)^2\right) \quad (3.4)$$

where we choose the states for which $\langle X_i \rangle = 0$.

According to the value of τ we distinguish two kinds of subalgebra. For $\tau = -1$, the deformed algebra is characterized by the presence of a nonzero minimum uncertainty in momentum and it is called anti-deSitter model. For simplicity, we assume isotropic uncertainties $X_i = X$ and this allows us to write the minimal uncertainty for the momentum in

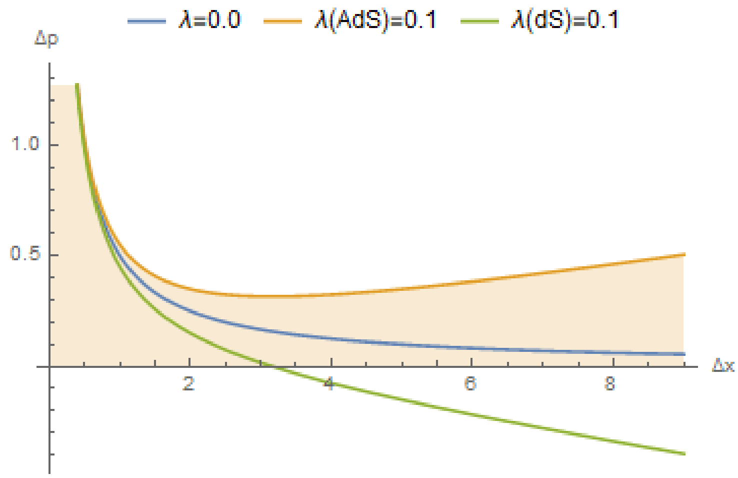

Figure 3.1: Graphic of HUP and EUP in both dS and AdS Cases

anti-deSitter model:

$$(\Delta P_i)_{\text{min}} = \hbar\sqrt{\tau\lambda} \tag{3.5}$$

For de Sitter model $\tau = +1$, the relation 1.204 does not imply a non-zero minimal value for momentum uncertainties.

This is shown in($Figure$ 3.1),where the Heisenberg uncertainty relations are plotted according to the modified relation found in 3.4 . The hatched region in ($Figure$ 3.1) is the forbidden area for position and momentum measurements in Anti-deSitter space.

From now on, we will employ the noncommutative operators X_i and P_i satisfying the modified algebra 1.204 which gives rise to rescaled uncertainty relation 1.204 in momentum space. In order to study the exact solutions of the deformed Schrödinger equation, we represent these operators as functions of the operators $x_i$and p_i that satisfy the ordinary canonical commutation relations; This is done thanks to the following transformations:

$$X_i = \frac{x_i}{\sqrt{1+\tau\lambda r^2}} P_i = -i\hbar\sqrt{1+\tau\lambda r^2}\partial_{x_i} \tag{3.6a}$$

$$P_i = -i\hbar\sqrt{1+\tau\lambda r^2}\partial_{x_i} \tag{3.6b}$$

When $\tau = -1$, the variable r varies in the domain $\left]-1/\sqrt{\lambda}, 1/\sqrt{\lambda}\right[$.

3.2 2D Schrödinger Equation of N-C Potential in (dS and AdS) Spaces

In this section, we clarify the effect of deformed space on the energy eigenvalues and eigenfunctions of a non relativistic system in presence of non-central potential $V(r, \theta)$ which is given by

$$V(r,\theta)=\mu\left(V(r)+\frac{f(\theta)}{r^2}\right) \tag{3.7}$$

When $V(r)$ take the form $\left(-\frac{H}{r}+\frac{D_r}{r^2}\ ,\ -\frac{H}{r}\ ,\ kr^2+\frac{D_r}{r^2}\ ,\ kr^2\right)$,and $f(\theta)$ is given in $(Table1)$

We consider the following stationary Schrödinger equation with a non-central potential-type interaction

$$\left[\frac{\mathbf{p}^2}{2\mu}+\mu\left(V(r)+\frac{f(\theta)}{\boldsymbol{r}^2}\right)\right]\psi(r,\theta)=E\psi(r,\theta) \tag{3.8}$$

In order to include the effect of EUP on the above Schrödinger equation, we use the transformations 3.6a and 3.6bto obtain the deformed Schrödinger equation

The vector position transform as

$$\boldsymbol{r}=\frac{r}{\sqrt{1+\tau\lambda r^2}} \tag{3.9}$$

So

$$\boldsymbol{r}^2=\frac{r^2}{1+\tau\lambda r^2} \tag{3.10}$$

The momentum transform as

$$\boldsymbol{p}^2=\left(\sqrt{1+\tau\lambda r^2}p\right)^2=\left(1+\tau\lambda r^2\right)p^2+\tau\lambda rp \tag{3.11}$$

The Schrödinger equation in deformed space is written as

$$\left[\frac{\left(\sqrt{1+\tau\lambda r^2}p\right)^2}{2\mu}+\mu\left(V(\frac{r}{\sqrt{1+\tau\lambda r^2}})+\frac{\left(1+\tau\lambda r^2\right)f(\theta)}{r^2}\right)\right]\psi(r,\theta)=E\psi(r,\theta) \tag{3.12}$$

We substitute the equations 3.9 ,3.10 and 3.11 in equation 3.12we find

$$\left[\frac{1}{2\mu}\left(\left(1+\tau\lambda r^2\right)p^2+\tau\lambda rp\right)+\mu\left(V(\frac{r}{\sqrt{1+\tau\lambda r^2}})+\frac{\left(1+\tau\lambda r^2\right)f(\theta)}{r^2}\right)\right]\psi(r,\theta)=E\psi(r,\theta) \tag{3.13}$$

Using the polar coordinates $0\leq r<\infty$ and $0\leq\theta\leq 2\pi$, and we write the Schrödinger equation 3.13 as follows

$$\begin{aligned}&\left[-\frac{\hbar^2}{2\mu}\left[\left(1+\tau\lambda r^2\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)+\tau\lambda r\frac{\partial}{\partial r}\right]+\right.\\&\left.\mu\left(V(\frac{r}{\sqrt{1+\tau\lambda r^2}})+\frac{\left(1+\tau\lambda r^2\right)f(\theta)}{r^2}\right)\right]\psi=E\psi\end{aligned} \tag{3.14}$$

We put the equation in the more convenient following form:

$$\left[\left(\sqrt{1+\tau\lambda r^2}\frac{\partial}{\partial r}\right)^2+\frac{(1+\tau\lambda r^2)}{r}\frac{\partial}{\partial r}-\frac{2\mu^2}{\hbar^2}V(\frac{r}{\sqrt{1+\tau\lambda r^2}})\right.$$
$$\left.+\frac{(1+\tau\lambda r^2)}{r^2}\left(\frac{\partial^2}{\partial\theta^2}-\frac{2\mu^2}{\hbar^2}f\left(\theta\right)\right)\right]\psi=-\frac{2\mu E}{\hbar^2}\psi \tag{3.15}$$

We write the solution as $\psi(r,\theta)=R(r)\Theta(\theta)$, to get two separate equations as in previous section

The angular equation is

$$\left(\frac{d^2}{d\theta^2}-\frac{2\mu^2}{\hbar^2}f\left(\theta\right)-E_\theta\right)\Theta(\theta)=0 \tag{3.16}$$

The radial equation is

$$\left[\left(\sqrt{1+\tau\lambda r^2}\frac{d}{dr}\right)^2+\frac{(1+\tau\lambda r^2)}{r}\frac{d}{dr}+\frac{(1+\tau\lambda r^2)}{r^2}E_\theta-\frac{2\mu^2}{\hbar^2}V(\frac{r}{\sqrt{1+\tau\lambda r^2}})+\frac{2\mu E}{\hbar^2}\right]R(r)=0 \tag{3.17}$$

E_θ is separation constant.

Now we have to solve this equations

3.3 Non-Relativistic Solutions of N-C Potentials in 2D Deformed Space

Case1 $V_1\left(r,\theta\right)=\mu\left[-\frac{H}{r}+\frac{D_r}{r^2}+\frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cos\theta\right)\right]$

Solution of Angular Equation We note that this equation is the same of the case of the ordinary space of the first chapter ,then the constant of separation and the angular part of wave function are deduced by the same manner of the first chapter

The angular part of wave functions and the constant of separation are appear in equations 1.34,1.34

Solution of Radial Equation

The Nikiforov-Uvarov (NU) Method Nikiforov-Uvarov (NU) method was developed basically on the hypergeometric differential equation. The formulas used in NU method reduce the second order differential equations to the hypergeometric type with an appropriate coordinate transformation $s=s\left(x\right)$:

$$\psi''\left(s\right)+\frac{\tilde{\tau}\left(s\right)}{\sigma\left(s\right)}\psi'\left(s\right)+\frac{\tilde{\sigma}\left(s\right)}{\sigma^2\left(s\right)}\psi\left(s\right)=0 \tag{3.18}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials of the second degree at most and the degree of the polynomial $\widetilde{\tau}(s)$ is strictly less than 2 [114][115]. If we take the following factorization:

$$\psi(s) = \phi(s)\, y(s) \tag{3.19}$$

the equation 3.19 becomes [115]:

$$\sigma(s)\, y''(s) + \tau(s)\, y'(s) + \Lambda y(s) = 0 \tag{3.20}$$

where:

$$\pi(s) = \sigma(s)\frac{d}{ds}(\ln\phi(s)) \quad \text{and} \quad \tau(s) = \tilde{\tau}(s) + 2\pi(s) \tag{3.21}$$

Λ is defined as:

$$\Lambda_n + n\tau' + \frac{n(n-1)\sigma''}{2} = 0, \qquad n_r = 0, 1, 2, ... \tag{3.22}$$

And the energy eigenvalues are calculated from the above equation. We first have to determine $\pi(s)$ and Λ by defining:

$$k = \Lambda - \pi'(s) \tag{3.23}$$

Solving the quadratic equation for $\pi(s)$ with 3.23 , we get

$$\pi(s) = \left(\frac{\sigma' - \widetilde{\tau}}{2}\right) \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + \sigma k} \tag{3.24}$$

Here, $\pi(s)$ is a polynomial with s as the parameter and the prime denotes the first .

One has to note that the determination of k is the essential point in the calculation of $\pi(s)$ and it is simply defined by stating that the expression under the square root in 3.21 must be a square of a polynomial; This gives us a general quadratic equation for k.

To determine the polynomial solutions $y_n(s)$, we use 3.20 and the Rodrigues relation:

$$y_n(s) = \frac{C_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\,\rho(s)\right] \tag{3.25}$$

where C_n is normalizable constant and the weight function $\rho(s)$ satisfies the following relation:

$$\frac{d}{ds}\left[\sigma(s)\,\rho(s)\right] = \tau(s)\,\rho(s) \tag{3.26}$$

This last equation refers to the classical orthogonal polynomials that have many important properties and especially orthogonality defined by:

$$\int_a^b y_n(s)\, y_m(s)\, \rho(s)\, ds = 0 if m \neq n$$

So in this case the radial equation is

$$\Bigg[\left(\sqrt{1+\tau\lambda r^2}\frac{d}{dr}\right)^2+\frac{(1+\tau\lambda r^2)}{r}\frac{d}{dr}+\frac{(1+\tau\lambda r^2)}{r^2}E_\theta-\frac{2\mu^2}{\hbar^2}\left(-\frac{\sqrt{1+\tau\lambda r^2}H}{r}+\frac{(1+\tau\lambda r^2)D_r}{r^2}\right)+\frac{2\mu E}{\hbar^2}\Bigg]R(r)=0 \tag{3.27}$$

After sum simplification we get

$$\Bigg[\left(\sqrt{1+\tau\lambda r^2}\frac{d}{dr}\right)^2+\frac{(1+\tau\lambda r^2)}{r}\frac{d}{dr}+\frac{(1+\tau\lambda r^2)}{r^2}\left(E_\theta-\frac{2\mu^2 D_r}{\hbar^2}\right)+\frac{2\mu^2 H}{\hbar^2}\frac{\sqrt{1+\tau\lambda r^2}}{r}+\frac{2\mu E}{\hbar^2}\Bigg]R(r)=0 \tag{3.28}$$

In order to solve this radial equation we use the Nikiforov–Uvarov method ,when we use the following transformations

$$s=\frac{\sqrt{1+\tau\lambda r^2}}{\sqrt{\lambda}r}\Longrightarrow s^2=\frac{1+\tau\lambda r^2}{\lambda r^2}\Longrightarrow s^2-\tau=\frac{1}{\lambda r^2}\Longrightarrow r=\frac{1}{\sqrt{\lambda}\sqrt{(s^2-\tau)}}\Longrightarrow$$
$$\frac{ds}{dr}=\left(\frac{\tau\lambda r}{\sqrt{1+\tau\lambda r^2}}\sqrt{\lambda}r-\sqrt{\lambda}\sqrt{1+\tau\lambda r^2}\right)\frac{1}{\lambda r^2}=\left(\frac{-\sqrt{\lambda}}{\sqrt{1+\tau\lambda r^2}}\right)\frac{1}{\lambda r^2} \tag{3.29}$$

By using this transformation we can writ the following derivative in terms of new variable s

$$\frac{(1+\tau\lambda r^2)}{r}\frac{d}{dr}=\frac{(1+\tau\lambda r^2)}{r}\left(\frac{-\sqrt{\lambda}}{\sqrt{1+\tau\lambda r^2}}\right)\frac{1}{\lambda r^2}=-\lambda s\left(s^2-\tau\right)\frac{d}{ds} \tag{3.30}$$

and

$$\sqrt{1+\tau\lambda r^2}\frac{d}{dr}=-\sqrt{\lambda}\left(s^2-\tau\right)\frac{d}{ds} \tag{3.31}$$

We substitute this derivatives in equation 3.28 we find

$$\Bigg[\left(-\sqrt{\lambda}\left(s^2-\tau\right)\frac{d}{ds}\right)^2-\lambda s\left(s^2-\tau\right)\frac{d}{ds}+\lambda s^2\left(E_\theta-\frac{2\mu^2 D_r}{\hbar^2}\right)+\frac{2\mu^2 H}{\hbar^2}\sqrt{\lambda}s+\frac{2\mu E}{\hbar^2}\Bigg]R(s)=0 \tag{3.32}$$

We divide by λ ,then, the equation 3.32 becomes as

$$\left[\left(\tau-s^2\right)^2\frac{d^2}{ds^2}-s\left(\tau-s^2\right)\frac{d}{ds}+\left(E_\theta-\frac{2\mu^2 D_r}{\hbar^2}\right)s^2+\eta s+\varepsilon\right]R_{1,2}(s)=0 \tag{3.33}$$

Where

$$\eta=\frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}},\quad \varepsilon=\frac{2\mu E}{\lambda\hbar^2} \tag{3.34}$$

We divide the last equation by $(\tau - s^2)^2$ that give arise

$$\left[\frac{d^2}{ds^2} - \frac{s}{(\tau - s^2)}\frac{d}{ds} + \frac{1}{(\tau - s^2)^2}\left(\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2}\right)s^2 + \eta s + \varepsilon\right)\right] R_{1,2}(s) = 0 \tag{3.35}$$

Solution of the Radial Equation in de Sitter Space $(\tau = +1)$ This case is represented by the equation 3.35 with $(\tau = +1)$ as

$$\left[\frac{d^2}{ds^2} - \frac{s}{(1 - s^2)}\frac{d}{ds} + \frac{1}{(1 - s^2)^2}\left(\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2}\right)s^2 + \eta s + \varepsilon\right)\right] R_{1,2}(s) = 0 \tag{3.36}$$

To determine polynomials we compare equation 3.36 with equation 3.18,so

$$\sigma(s) = \left(1 - s^2\right), \quad \tilde{\tau}(s) = -s \quad \text{and} \quad \tilde{\sigma}(s) = \left(E_\theta - \frac{2\mu q D_r}{4\pi\epsilon_0 \hbar^2}\right)s^2 + \eta s + \varepsilon \tag{3.37}$$

Substituting them into equation 3.24: $\pi(s) = \left(\frac{\sigma' - \tilde{\tau}}{2}\right) \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + \sigma k}$ we obtain

$$\pi(s) = \frac{-s}{2} \pm \sqrt{\left(\frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - k\right)s^2 - \eta s + k - \varepsilon} \tag{3.38}$$

The value of k is obtained from the condition that quadratic expression under the square root in 3.38 has to be completely square of first degree of polynomial

$$\pi(s) = \frac{-s}{2} \pm \sqrt{\left(\frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - k\right)}(s - s_0) \tag{3.39}$$

And

$$s_0 = \frac{\eta}{2\left(\frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - k\right)} \tag{3.40}$$

Therefore the discriminate of the quadratic expression under the square root that has to be zero is given as

$$\eta^2 - 4\left(\frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - k\right)(k - \varepsilon) = 0 \tag{3.41}$$

We writ the last equation as algebraic equation of second degree with respect to k

$$4k^2 - \left(1 - 4\left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) + 4\varepsilon\right)k + \left(1 - 4\left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right)\right)\varepsilon + \eta^2 = 0 \tag{3.42}$$

Now to find k we have to solve this equation ,the discriminate of this equation is $\triangle$

$$\triangle = \left(\varepsilon - \frac{1}{4} + \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right)\right)^2 - \eta^2 \tag{3.43}$$

So we have two solution for the equation 3.42 k_1 and k_2

$$k = \begin{cases} k_1 = \frac{1}{2}\left[\varepsilon + \frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) + \sqrt{\triangle}\right] \\ k_2 = \frac{1}{2}\left[\varepsilon + \frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - \sqrt{\triangle}\right] \end{cases} \tag{3.44}$$

We put

$$\delta_{1,2} = \sqrt{\frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - k_{1,2}} \tag{3.45}$$

From the equations 3.40 and 3.45 s_0 can be written as

$$s_0 = \frac{\eta}{2\left(\frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - k_{1,2}\right)} = \frac{\eta}{2\left(\delta_{1,2}\right)^2} \tag{3.46}$$

Substitute it in equation 3.39 ,then

$$\pi(s) = \begin{cases} \pi_{1,2} = \left(\frac{-1}{2} \pm \delta_1\right) s \pm \frac{\eta}{2\delta_1}, \text{ for } k_1 = \frac{1}{2}\left[\varepsilon + \frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) + \sqrt{\triangle}\right] \\ \pi_{3,4} = \left(\frac{-1}{2} \pm \delta_2\right) s \pm \frac{\eta}{2\delta_2}, \text{ for } k_2 = \frac{1}{2}\left[\varepsilon + \frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - \sqrt{\triangle}\right] \end{cases} \tag{3.47}$$

Here, we choose k_1 and π_1 witch give us the limit of ordinary space , and use the relation $\tau(s) = \tilde{\tau}(s) + 2\pi(s)$,then

$$\tau(s) = -s + 2\left[\left(\frac{-1}{2} + \delta_1\right) s + \frac{\eta}{2\delta_1}\right] = 2\left(\delta_1 - 1\right) s + \frac{\eta}{\delta_1} \tag{3.48}$$

From equation 3.23, we calculate

$$k_1 = \Lambda - \pi_1^{\prime}(s) \Longrightarrow \Lambda = k_1 + \pi_1^{\prime}(s) = k_1 - \frac{1}{2} + \delta_1 \tag{3.49}$$

In other hand from equations 3.22, 3.48 and the relations $\sigma(s) = (1 - s^2)$,we have

$$\Lambda = -n_r \tau^{\prime} - \frac{n_r(n_r - 1)\sigma^{\prime\prime}}{2} = -n_r(2)(\delta_1 - 1) - \frac{n_r(n_r - 1)(-2)}{2} = n_r(n_r + 1 - 2\delta_1), \;\; n_r = 0, 1, 2, ... \tag{3.50}$$

We not that equation 3.49 and equation 3.50 are equal that means

$$k_1 - \frac{1}{2} + \delta_1 = n_r(n_r + 1 - 2\delta_1) \Longrightarrow k_1 = \frac{1}{2} - \delta_1(2n_r + 1) + n_r(n_r + 1) \tag{3.51}$$

Now we have substitute the expression of k_1 and $\delta_1$from equation 3.44 and equation 3.45 ,then we get a algebraic equation of second order for k_1,we have to solve it to find the energy

$$k_1^2+(2n_r^2+2n_r)k+n_r^2\left(n_r+1\right)^2-\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)\left(2n_r+1\right)^2=0 \tag{3.52}$$

We solve the last equation we find two solution

$$_1k_1=\frac{-(2n_r^2+2n_r)-2(2n_r+1)\sqrt{-\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)}}{2}=-n_r(n_r+1)-(2n_r+1)\sqrt{-\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)} \tag{3.53}$$

And

$$_2k_1=\frac{-(2n_r^2+2n_r)+2(2n_r+1)\sqrt{-\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)}}{2}=-n_r(n_r+1)+(2n_r+1)\sqrt{-\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)} \tag{3.54}$$

We choose the solution $_2k_1$ which give as the correct limit of energy

In other side we have $k_1=\frac{1}{2}\left[\varepsilon+\frac{1}{4}-\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)+\sqrt{\left(\varepsilon-\frac{1}{4}+\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)\right)^2-\eta^2}\right]$,so

$$\frac{1}{2}\left[\varepsilon+\frac{1}{4}-\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)+\sqrt{\left(\varepsilon-\frac{1}{4}+\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)\right)^2-\eta^2}\right]=$$
$$-n_r(n_r+1)-(2n_r+1)\sqrt{-\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)} \tag{3.55}$$

we solve the above equation to find and after we substitute by the expressions of $\varepsilon=\frac{2\mu E}{\lambda\hbar^2}$ and $\eta=\frac{2\mu^2H}{\hbar^2\sqrt{\lambda}}$,hence, the energy eigenvalues are found as

$$E_n=-2\frac{\mu^3H^2}{\hbar^2}\left(2n_r+2\sqrt{-E_\theta+\frac{2\mu D_r}{\hbar^2}}+1\right)^{-2}-$$
$$\frac{\lambda\hbar^2}{8\mu}\left[(2n_r+1)\left(2n_r+1+4\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)-1\right] \tag{3.56}$$

where n is the principal quantum number.

The wave function $R_n\left(x\right)$ is obtained from equation 3.19 by using $\phi\left(s\right)$ and $y_n\left(s\right)$ as follows. We first get $\pi\left(s\right)$ from equation 3.21

$$\pi\left(s\right)=\sigma\left(s\right)\frac{d}{ds}\left(\ln\phi\left(s\right)\right)\Longrightarrow\phi\left(s\right)=Exp\left(\int\frac{\pi\left(s\right)}{\sigma\left(s\right)}ds\right)\Longrightarrow\phi\left(s\right)=Exp\left(\int\frac{\pi\left(s\right)}{\sigma\left(s\right)}ds\right) \tag{3.57}$$

We substitute by the expression of $\pi\left(s\right)=\pi_1\left(s\right)$ equation 3.47 and $\sigma\left(s\right)$ equation 3.37 we find

$$\phi(s) = Exp\left(\int \frac{\left(\frac{-1}{2}+\delta_1\right)s+\frac{\eta}{2\delta_1}}{(1-s^2)}ds\right) \Longrightarrow$$
$$\phi(s) = Exp\left(\left(\frac{-1}{2}+\delta_1\right)\int \frac{s}{(1-s^2)}ds+\frac{\eta}{2\delta_1}\int \frac{1}{(1-s^2)}ds\right)$$
$$= Exp\left(\int \frac{\left(\frac{-1}{2}+\delta_1\right)s+\frac{\eta}{2\delta_1}}{(1-s)(1+s)}ds\right) \tag{3.58}$$

After the calculation of the integral we obtain

$$\phi(s) = Exp\left(\ln(1+s)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)}(1-s)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)}\right) \tag{3.59}$$

So the function $\phi(s)$ is

$$\phi(s) = (1+s)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)}(1-s)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)} \tag{3.60}$$

We use 3.26 to find the weight function $\rho(s)$

$$\frac{d}{ds}\left[\sigma(s)\rho(s)\right] = \tau(s)\rho(s) \Longrightarrow \int \frac{d\rho(s)}{\rho(s)} = \int\left(\frac{\tau(s)}{\sigma(s)}-\frac{d\sigma(s)}{\sigma(s)ds}\right)ds \tag{3.61}$$

When we compute the integral we get

$$\ln\rho(s) = \int\left(\frac{\tau(s)}{\sigma(s)}-\frac{d\sigma(s)}{\sigma(s)ds}\right)ds \Longrightarrow \rho(s) = Exp\left[\int\left(\frac{\tau(s)}{\sigma(s)}-\frac{d\sigma(s)}{\sigma(s)ds}\right)ds\right] \tag{3.62}$$

We substitute by the expression of $\tau(s)$equation 3.48 and $\sigma(s)$ equation 3.37 we find

$$\rho(s) = Exp\left[\int\left(\frac{2\delta_1 s+\frac{\eta}{\delta_1}}{(1-s^2)}\right)ds\right] \Longrightarrow$$
$$\rho(s) = Exp\left[2\delta_1\int\left(\frac{s}{(1-s^2)}\right)ds+\frac{\eta}{\delta_1}\int\left(\frac{1}{(1-s^2)}\right)ds\right] \tag{3.63}$$

After the calculation of the integral we find

$$\rho(s) = Exp\left[\ln\left((1-s)^{-\delta_1-\frac{\eta}{2\delta_1}}(1+s)^{-\delta_1+\frac{\eta}{2\delta_1}}\right)\right] \tag{3.64}$$

So we have

$$\rho(s) = (1+s)^{\left(-\delta_1+\frac{\eta}{2\delta_1}\right)}(1-s)^{-\left(\delta_1+\frac{\eta}{2\delta_1}\right)} \tag{3.65}$$

the $y_n(s)$part is given by Rodrigues relation

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} \left[\left(1-s^2\right)^n \rho(s) \right] \tag{3.66}$$

We substitute by the expression of $\rho(s)$ from equation 3.65

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} \left[\left(1-s^2\right)^n (1+s)^{\left(-\delta_1+\frac{\eta}{2\delta_1}\right)} (1-s)^{-\left(\delta_1+\frac{\eta}{2\delta_1}\right)} \right] \tag{3.67}$$

$y_n(s)$ stands for the Jacobi polynomials as

$$y_n(s) \equiv P_n^{\left(-\delta_1+\frac{\eta}{2\delta_1}, -\delta_1-\frac{\eta}{2\delta_1}\right)}(s) \tag{3.68}$$

Hence, $R(s)$ can be deduce from the equation 3.20 $R(s) = \phi(s)\, y_n(s)$ written in the following form

$$R(s) = C_n (1-s)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)} (1+s)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)} P_n^{\left(-\delta_1+\frac{\eta}{2\delta_1}, -\delta_1-\frac{\eta}{2\delta_1}\right)}(s) \tag{3.69}$$

In terms of the variables r,we can now write the radial wave function $R(r)$ as follows:

$$R(s) = C_n \left(1 - \frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda} r}\right)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)} \left(1 + \frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda} r}\right)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)} P_n^{\left(-\delta_1+\frac{\eta}{2\delta_1}, -\delta_1-\frac{\eta}{2\delta_1}\right)}\left(\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda} r}\right) \tag{3.70}$$

Where $\delta_1 = \sqrt{\frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - -n_r(n_r+1) + (2n_r+1)\sqrt{-\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2}\right)}}, \eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$
C_n is a normalization constant

Solution of the Radial Equation in Anti de Sitter Space This case is represented by the equation 3.35 with $(\tau = -1)$ as

$$\left[\frac{d^2}{ds^2} + \frac{s}{(1+s^2)}\frac{d}{ds} + \frac{1}{(1+s^2)^2}\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2}\right)s^2 - \eta s + \varepsilon\right] R_{1,2}(s) = 0 \tag{3.71}$$

As the same way when comparing equation 3.71 with equation 3.18 , we determine polynomials as

$$\sigma(s) = \left(1+s^2\right), \quad \tilde{\tau}(s) = s \ \text{ and } \tilde{\sigma}(s) = \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right)s^2 - \eta s + \varepsilon \tag{3.72}$$

Substituting them into equation 3.24 we obtain

$$\pi(s) = \frac{s}{2} \pm \sqrt{\left(k + \frac{1}{4} - \left(E_{\theta}^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right)\right) s^2 + \eta s + k - \varepsilon} \tag{3.73}$$

The constant k is determined in the same way as in deSitter case. Therefore, we get:

$$\pi(s) = \begin{cases} \pi_{1,2} = \left(\frac{-1}{2} \pm \delta_1\right) s \pm \frac{\eta}{2\delta_1}, \text{ for } k_1 = \frac{1}{2}\left[\varepsilon + \frac{1}{4} - \left(E_{\theta}^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) + \sqrt{\Delta}\right] \\ \pi_{3,4} = \left(\frac{-1}{2} \pm \delta_2\right) s \pm \frac{\eta}{2\delta_2}, \text{ for } k_2 = \frac{1}{2}\left[\varepsilon + \frac{1}{4} - \left(E_{\theta}^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - \sqrt{\Delta}\right] \end{cases} \tag{3.74}$$

Where

$$\delta = \sqrt{\frac{1}{4} - \left(E_{\theta}^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) + k} \text{ and } \Delta = \left(\varepsilon + \frac{1}{4} - \left(E_{\theta}^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right)\right)^2 + \eta^2 \tag{3.75}$$

Here, we choose k_1 and $\pi_2$from equation.3.74 for the limit in ordinary space

$$\tau(s) = 2(1 - \delta_1) s + \frac{\eta}{\delta_1} \tag{3.76}$$

From Equation.3.38, we calculate

$$\Lambda = k_1 + \frac{1}{2} - \sqrt{\frac{1}{4} - \left(E_{\theta}^{(2m)} - \frac{2mqD_r}{4\pi\epsilon_0\hbar^2}\right) + k_1} = -n_r\left(n_r + 1 - 2\sqrt{\frac{1}{4} - \left(E_{\theta}^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) + k_1}\right) \tag{3.77}$$

Hence, the energy eigenvalues are found as

$$\begin{aligned} E_n &= -2\frac{\mu^3 H^2}{\hbar^2}\left(2n_r + 2\sqrt{-E_{\theta} + \frac{2\mu D_r}{\hbar^2}} + 1\right)^{-2} + \\ \frac{\lambda\hbar^2}{8\mu} &\left[(2n_r + 1)\left(2n_r + 1 + 4\sqrt{-E_{\theta}^{(2m)} + \frac{2\mu^2 D_r}{\hbar^2}}\right) - 1\right] \end{aligned} \tag{3.78}$$

We see that the energy spectrum in de Sitter space is smaller than the energy in anti-de Sitter space.

To deduce the complete expression of the wave functions$R(r)$, we use the relations of $\pi_2(s)$ as follows. We first get

$$\begin{gathered} \pi(s) = \pi_2(s) = \sigma(s)\frac{d}{ds}(\ln\phi(s)) \Longrightarrow \\ \phi(s) = Exp\left(\int \frac{\pi(s)}{\sigma(s)} ds\right) \Longrightarrow \phi(s) = Exp\left(\int \frac{\pi(s)}{\sigma(s)} ds\right) \end{gathered} \tag{3.79}$$

We substitute by the expression of $\pi_2(s)$ equation 3.74and $\sigma(s)$ equation 3.72we find

$$\phi(s) = Exp\left(\int \frac{\left(\frac{1}{2}+\delta_1\right)s + \frac{\eta}{2\delta_1}}{(1+s^2)} ds\right) \Longrightarrow$$
$$\phi(s) = Exp\left(\left(\frac{1}{2}+\delta_1\right)\int \frac{s}{(1+s^2)} ds + \frac{\eta}{2\delta_1}\int \frac{1}{(1+s^2)} ds\right) \tag{3.80}$$

After the calculation of the integral we obtain the function $\phi(s)$ as

$$\phi(s) = \left(1+s^2\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_1^{\prime}\right)} e^{\frac{\eta}{2\delta_1^{\prime}}\tan^{-1}(s)} \tag{3.81}$$

We use equation 3.62 to find the weight function $\rho(s)$ when substitute by the expression of $\tau(s)$ equation 3.76and $\sigma(s)$ equation 3.72we find

$$\ln\rho(s) = \int\left(\frac{\tau(s)}{\sigma(s)} - \frac{d\sigma(s)}{\sigma(s)\,ds}\right) ds \Longrightarrow \rho(s) = Exp\left[\int\left(\frac{2\left(1-\delta_1^{\prime}\right)s + \frac{\eta}{\delta_1^{\prime}}}{(1+s^2)} - \frac{2s}{(1+s^2)}\right) ds\right]$$
$$\Longrightarrow \rho(s) = Exp\left[-2\delta_1^{\prime}\int\left(\frac{s}{(1+s^2)}\right) ds + \frac{\eta}{\delta_1^{\prime}}\int\left(\frac{1}{(1+s^2)}\right) ds\right] \tag{3.82}$$

After the calculation of the integral we get

$$\rho(s) = \left(1+s^2\right)^{-\delta_1^{\prime}} e^{\frac{\eta}{\delta_1^{\prime}}\tan^{-1}(s)} \tag{3.83}$$

The $y_n(s)$ part is given by Rodrigues relation

And using Rodrigues formula expressed in.3.25,3.26 and $\sigma(s)$ from equation 3.72 , we find

$$y_n(s) = \frac{C_n}{\rho(s)}\frac{d^n}{ds^n}\left[\left(1+s^2\right)^n \rho(s)\right] \tag{3.84}$$

Where $\rho(s) = (1+s^2)^{-\delta_1^{\prime}} e^{\frac{\eta}{\delta_1^{\prime}}\tan^{-1}(s)}$ equation 3.84 stands for the Romanovski polynomials as

$$y_n(s) \equiv R_n^{\left(-\delta_1^{\prime}, \frac{\eta}{\delta_1^{\prime}}\right)}(s) = \frac{C_n}{(1+s^2)^{-\delta_1^{\prime}} e^{\frac{\eta}{\delta_1^{\prime}}\tan^{-1}(s)}}\frac{d^n}{ds^n}\left[\left(1+s^2\right)^{n-\delta_1^{\prime}} e^{\frac{\eta}{\delta_1^{\prime}}\tan^{-1}(s)}\right] \tag{3.85}$$

Hence, $R(s)$ can be written in the following form

$$R_2(s) = C_n\left(1+s^2\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_1\right)} e^{\frac{\eta}{2\delta_1}\tan^{-1}(s)} R_n^{\left(-\delta_1, \frac{\eta}{\delta_1}\right)}(s) \tag{3.86}$$

In terms of the variables r,we can now write the radial wave function $R(r)$ as follows:$\frac{\sqrt{1-\lambda r^2}}{\sqrt{\lambda}r}$

$$R_2(s) = C_n\left(1+s^2\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_1\right)} e^{\frac{\eta}{2\delta_1}\tan^{-1}(s)} R_n^{\left(-\delta_1, \frac{\eta}{\delta_1}\right)}(s) \tag{3.87}$$

C_n is the normalization constant

In terms of the variables r,we can now write the radial wave function $R(r)$ as follows:

$$R(r) = C_n \left(1 + \frac{1 - \lambda r^2}{\lambda r}\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_1\right)} e^{\frac{\eta}{2\delta_1}\tan^{-1}(s)} R_n^{\left(-\delta_1, \frac{\eta}{\delta_1}\right)} \left(\frac{\sqrt{1-\lambda r^2}}{\sqrt{\lambda} r}\right) \tag{3.88}$$

Where $\delta_1 = \sqrt{\frac{1}{4} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - n_r(n_r+1) + (2n_r+1)\sqrt{-\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2}\right)}}, \eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$

Non-Relativistic Energy and Wave Function

de Sitter Space We substitute the constant of separation from equation 1.34 in the energy expression 3.56 ,we get the deformed energy as follows:

$$E_n = -2\frac{\mu^3 H^2}{\hbar^2}\left(2n_r + 2\sqrt{\frac{1}{4}c_{2m}(2\alpha) + \frac{2\mu^2 D_r}{\hbar^2}} + 1\right)^{-2} - \\ \frac{\lambda\hbar^2}{8\mu}\left[(2n_r+1)\left(2n_r + 1 + 4\sqrt{\frac{1}{4}c_{2m}(2\alpha) + \frac{2\mu^2 D_r}{\hbar^2}}\right) - 1\right] \tag{3.89}$$

We deduce the wave function of our system $\psi(r,\theta) = \psi(r,\theta) = R(r)\Theta(\theta)$ from the angular part 1.35 and radial part 3.70

$$\psi(r,\theta) = N\left(1 - \frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda} r}\right)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)}\left(1 + \frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda} r}\right)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)} \\ P_n^{\left(-\delta_1+\frac{\eta}{2\delta_1}, -\delta_1-\frac{\eta}{2\delta_1}\right)}\left(\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda} r}\right) \times \Theta(\theta) \tag{3.90}$$

Where $\delta_1 = \sqrt{\frac{1}{4} + \left(\frac{1}{4}c_{2m}(2\alpha) + \frac{2\mu^2 D_r}{\hbar^2}\right) + n_r(n_r+1) - (2n_r+1)\sqrt{\left(\frac{1}{4}c_{2m}(2\alpha) + \frac{2\mu^2 D_r}{\hbar^2}\right)}},$

and $\eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$

$c_{2m}(p)$ Mathieu characteristic values and $\Theta(\theta)$ is Mathieufunction

Anti de Sitter Space We substitute the constant of separation from equation 1.34 in the energy expression 3.78 ,we get the deformed energy as follows:

$$
\begin{aligned}
E_n = &-2\frac{\mu^3 H^2}{\hbar^2}\left(2n_r + 2\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{2\mu^2 D_r}{\hbar^2}} + 1\right)^{-2} + \\
&\frac{\lambda\hbar^2}{8\mu}\left[\left(2n_r + 1\right)\left(2n_r + 1 + 4\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{2\mu^2 D_r}{\hbar^2}}\right) - 1\right]
\end{aligned}
\tag{3.91}
$$

We deduce the wave function of our system $\psi\left(r,\theta\right) = \psi(r,\theta) = R(r)\Theta(\theta)$ from the angular part 1.35 and radial part 3.88

$$
\psi(r,\theta) = N\left(1 + \frac{1-\lambda r^2}{\lambda r}\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_1\right)} e^{\frac{\eta}{2\delta_1}\tan^{-1}(s)} R_n^{\left(-\delta_1,\frac{\eta}{\delta_1}\right)}\left(\frac{\sqrt{1-\lambda r^2}}{\sqrt{\lambda}r}\right) \times \Theta(\theta) \tag{3.92}
$$

Where $\delta_1 = \sqrt{\frac{1}{4} + \left(\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{2\mu^2 D_r}{\hbar^2}\right) + n_r(n_r+1) - (2n_r+1)\sqrt{\left(\frac{1}{4}c_{2m}\left(2\alpha\right) + \frac{2\mu^2 D_r}{\hbar^2}\right)}}$,
and $\eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$
$c_{2m}(p)$ Mathieu characteristic values and $\Theta(\theta)$ is Mathieufunction

For the potential The potential $V_2\left(r,\theta\right) = \mu\left[-\frac{H}{r} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cos\theta\right)\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_1\left(r,\theta\right)$ when we put $D_r \longrightarrow 0$ so

de Sitter Space The deformed energy is

$$
\begin{aligned}
E_n = &-2\frac{\mu^3 H^2}{\hbar^2}\left(2n_r + 2\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)} + 1\right)^{-2} - \\
&\frac{\lambda\hbar^2}{8\mu}\left[\left(2n_r + 1\right)\left(2n_r + 1 + 4\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)}\right) - 1\right]
\end{aligned}
\tag{3.93}
$$

The deformed wave function is

$$
\begin{aligned}
\psi(r,\theta) \quad = \quad & N\left(1 - \frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)}\left(1 + \frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)} \\
& P_n^{\left(-\delta_1+\frac{\eta}{2\delta_1},-\delta_1-\frac{\eta}{2\delta_1}\right)}\left(\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right) \times \Theta(\theta)
\end{aligned}
\tag{3.94}
$$

Where $\delta_1 = \sqrt{\frac{1}{4} + \left(\frac{1}{4}c_{2m}\left(2\alpha\right)\right) + n_r(n_r+1) - (2n_r+1)\sqrt{\left(\frac{1}{4}c_{2m}\left(2\alpha\right)\right)}}$,and $\eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$,
$c_{2m}(p)$ Mathieu characteristic values and $\Theta(\theta)$ is Mathieufunction

Anti de Sitter Space The deformed energy is

$$E_n = -2\frac{\mu^3 H^2}{\hbar^2}\left(2n_r + 2\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)+1}\right)^{-2} + \frac{\lambda\hbar^2}{8\mu}\left[(2n_r+1)\left(2n_r+1+4\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)}\right)-1\right] \tag{3.95}$$

The deformed wave function is

$$\psi(r,\theta) = N\left(1+\frac{1-\lambda r^2}{\lambda r}\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_1\right)} e^{\frac{\eta}{2\delta_1}\tan^{-1}(s)} R_n^{\left(-\delta_1,\frac{\eta}{\delta_1}\right)}\left(\frac{\sqrt{1-\lambda r^2}}{\sqrt{\lambda}r}\right)\times\Theta(\theta) \tag{3.96}$$

Where $\delta_1 = \sqrt{\frac{1}{4}+\left(\frac{1}{4}c_{2m}\left(2\alpha\right)\right)+n_r(n_r+1)-(2n_r+1)\sqrt{\left(\frac{1}{4}c_{2m}\left(2\alpha\right)\right)}}$,and $\eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$, $c_{2m}(p)$ Mathieu characteristic values and $\Theta(\theta)$ is Mathieu function

In order to show the effects of the deformed Heisenberg algebra leading to EUP on the bound states of the Coulomb potential in nonrelativistic quantum mechanics systems, we plot, as an example, the energy levels of the s-states $E_{n,0}$ versus the deformation parameters λ for different values of n. (We use the Hartree atomic units.).According to the results shown in $(Figures 3.1, ..., 3.5)$ and to the expression of energies 3.91, it is clear that the deformation increases the energies in AdS case and thus decreases the binding energies of the states. We thus arrive at a critical point where the value of the deformation parameter cancels the bound state or $E_{n,0} = 0$:

$$\lambda_c(n,m) = \frac{16\left(2\left(n-m\right)+2\sqrt{\frac{1}{4}c_{2m}\left(4D_\theta\right)+2D_r}+1\right)^{-2}}{\left[(2n-2m+1)\left(2n-2m+1+4\sqrt{\frac{1}{4}c_{2m}\left(4D_\theta\right)+2D_r}\right)-1\right]} \tag{3.97}$$

This critical value of the spatial deformation parameter can be interpreted as a resonance point because the corresponding state of the atomic system ionizes.We give in Table 1 some

critical values $\lambda_c(n,m)$ corresponding to the first levels in $(Table\ 3.1)$ for the Colombian ,$(Table\ 3.2)$ for the Kratzer +dipole (ce solution) $(Table\ 3.3)$ for the Kratzer +dipole (se solution)

Note from 3.89 that this is not the case for dS space because the deformation increases the bonding of atomic states and so no ionization effect occurs here.$(Figures 3.1, ..., 3.5)$ and the expression of the dS energies 3.89 show that the deformation can reverse the order of energy levels since the correction depends on the main quantum number. If we take the level $n = 4$ as an example, we see that it decreases faster than the third level and therefore it becomes

λ_C	m=0	m=1	m=2	m=3	m=4
n=1	0.22	\	\	\	\
n=2	0.0266	0.032	\	\	\
n=3	0.0068	0.0074	0.0102	\	\
n=4	0.0024	0.00259	0.00308	0.0044	\
n=5	0.0011	0.00113	0.00127	0.00157	0.00236

Table 3.1: Critical values for the levels n = 2, 3, 4 and 5 in AdS case for the 2D Colombian potential

λ_C	m=0	m=1	m=2	m=3	m=4
n=1	0.0600	\	\	\	\
n=2	0.0116	0.0257	\	\	\
n=3	0.0036	0.0063	0.0132	\	\
n=4	0.0015	0.0022	0.0038	0.0068	\
n=5	0.0007	0.0010	0.0015	0.0022	0.0038

Table 3.2: Critical values for the levels n = 2, 3, 4 and 5 in AdS case for the 2D kratzer +dipole(ce solution) potential

λ_C	m=0	m=1	m=2	m=3	m=4
n=1	0.5684	\	\	\	\
n=2	0.2393	0.4424	\	\	\
n=3	0.1318	0.1980	0.3594	\	\
n=4	0.0834	0.1133	0.1684	0.2896	\
n=5	0.0575	0.0735	0.0992	0.1415	0.2404

Table 3.3: Critical values for the levels n = 2, 3, 4 and 5 in AdS case for the 2D kratzer +dipole(se solution) potential

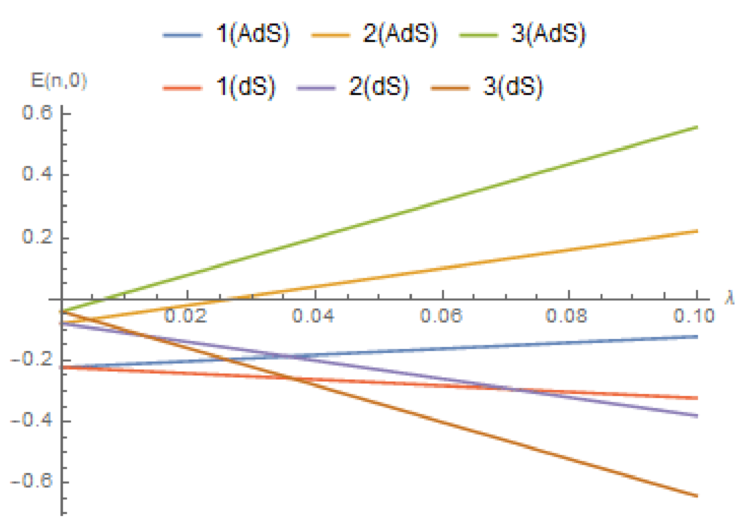

Figure 3.2: $E_{n;0}(\lambda)$ of (2D Colombian potential)for $n = 1, 2$ and 3in dS and AdS cases

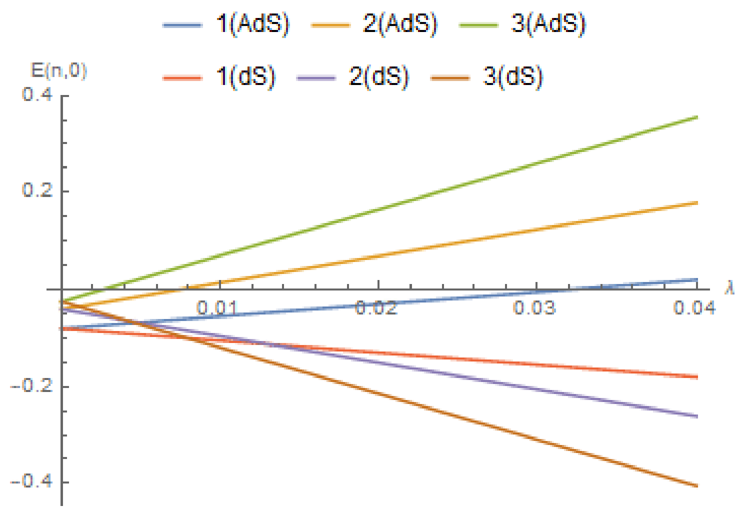

Figure 3.3: $E_{n;0}(\lambda)$ of (2D Kratzer potential) for $n = 1, 2$ and 3 in dS and AdS cases

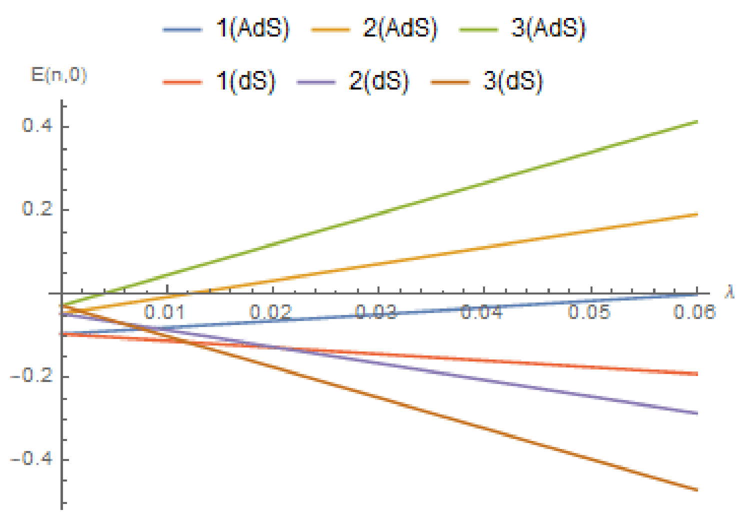

Figure 3.4: $E_{n,0}(\lambda)$ of 2D (Kratzer +dipole) potential (*ce solutions*) for $n = 1, 2$ and 3 in dS and AdS cases

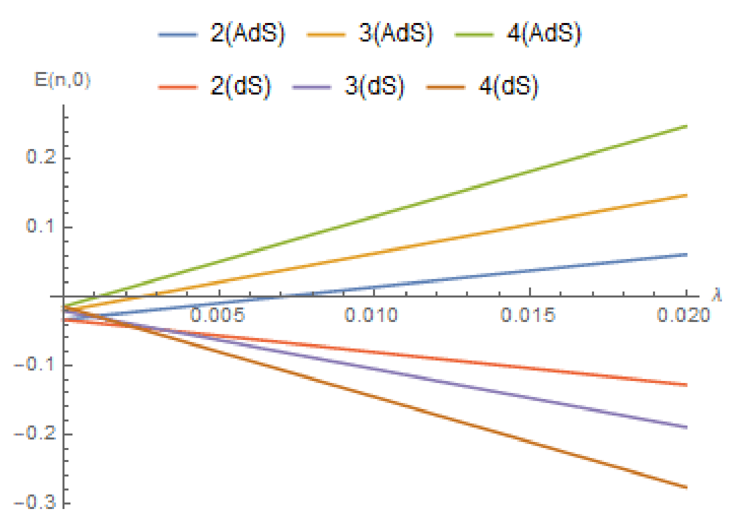

Figure 3.5: $E_{n,1}(\lambda)$ of 2D (Kratzer +dipole) potential (*se solutions*) for $n = 2, 3$ and 4 in dS and AdS cases

λ_f	m=0	m=1	m=2	m=3	m=4
n=1	0.7777	\	\	\	\
n=2	0.3066	0.368	\	\	\
n=3	0.1598	0.1743	0.2397	\	\
n=4	0.0975	0.1026	0.1219	0.1773	\
n=5	0.0655	0.0678	0.0756	0.0936	0.1404

Table 3.4: Critical values for the levels n = 2, 3,4 and 5 in dS case for the 2D colombian potential

lower. Then, it continues to decrease until it becomes lower than the second level, which will no longer be the fundamental one. The value of λ_f that causes this inversion between the upper levels and the fundamental one is calculated from 3.89

$$\lambda_f(n,m) = \frac{8\left(2(n-m)+\sqrt{\frac{1}{4}c_{2m}(4D_\theta)+2D_r}+1\right)^2-16}{\left(2(n-m)+\sqrt{\frac{1}{4}c_{2m}(4D_\theta)+2D_r}+1\right)^2} \times \frac{1}{\left[(2n-2m+1)\left(2n-2m+1+4\sqrt{\frac{1}{4}c_{2m}(4D_\theta)+2D_r}\right)-1\right]} \tag{3.98}$$

In ($Table$ 3.4), we give some numerical values of $\lambda_f(n,m)$.

Case 2: $V_3(r,\theta) = \mu\left[kr^2 + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos\theta)\right]$

Solution of Angular Equation The angular wave functions and constant of separation appear in equations 1.34 and 1.35

Solution of Radial Equation So in this case the radial equation is

$$\left[(1+\tau\lambda r^2)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)+\tau\lambda r\frac{d}{dr}+\frac{(1+\tau\lambda r^2)}{r^2}E_\theta - \frac{2\mu^2}{\hbar^2}\left(K\frac{r^2}{1+\tau\lambda r^2}+\frac{(1+\tau\lambda r^2)D_r}{r^2}\right)+\frac{2\mu E}{\hbar^2}\right]R(r)=0 \tag{3.99}$$

After some simplification we get

$$\left[(1+\tau\lambda r^2)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)+\tau\lambda r\frac{d}{dr}+\frac{(1+\tau\lambda r^2)}{r^2}\left(E_\theta-\frac{2\mu^2 D_r}{\hbar^2}\right) - \frac{2\mu^2 K}{\hbar^2}\frac{r^2}{1+\tau\lambda r^2}+\frac{2\mu E}{\hbar^2}\right]R(r)=0 \tag{3.100}$$

In order to solve this radial equation we use the following transformations

$$y=\sqrt{1+\tau\lambda r^{2}}\Longrightarrow r^{2}=\frac{y^{2}-1}{\tau\lambda} \tag{3.101}$$

We have to calculate the derivatives with respect to a new variable y,the first derivative is

$$\frac{d}{dr}=\frac{dy}{dr}\frac{d}{dy}=\frac{\tau\lambda r}{\sqrt{1+\tau\lambda r^{2}}}\frac{d}{dy}=\frac{\sqrt{\tau\lambda}\sqrt{y^{2}-1}}{y}\frac{d}{dy} \tag{3.102}$$

The second derivative is

$$\frac{d^{2}}{dr^{2}}=\frac{d}{dr}\left(\frac{\tau\lambda r}{\sqrt{1+\tau\lambda r^{2}}}\frac{d}{dy}\right)=\frac{\tau\lambda}{(1+\tau\lambda r^{2})^{\frac{3}{2}}}\frac{d}{dy}+\left(\frac{\tau\lambda r}{\sqrt{1+\tau\lambda r^{2}}}\right)^{2}\frac{d^{2}}{dy^{2}} \tag{3.103}$$

When we substitute the expression of r by y we find

$$\frac{d^{2}}{dr^{2}}=\frac{d}{dr}\left(\frac{\tau\lambda r}{\sqrt{1+\tau\lambda r^{2}}}\frac{d}{dy}\right)=\frac{\tau\lambda}{y^{3}}\frac{d}{dy}+\frac{\tau\lambda\left(y^{2}-1\right)}{y^{2}}\frac{d^{2}}{dy^{2}} \tag{3.104}$$

by using the derivatives of equations 3.102 and 3.103 the equation 3.100 becomes

$$\left[\left(y^{2}-1\right)\frac{d^{2}}{dy^{2}}+2y\frac{d}{dy}+\frac{y^{2}}{\left(y^{2}-1\right)}\left(E_{\theta}-\frac{2\mu^{2}D_{r}}{\hbar^{2}}\right)-\frac{2\mu^{2}K}{\hbar^{2}}\frac{\left(y^{2}-1\right)}{\tau^{2}\lambda^{2}y^{2}}+\frac{2\mu E}{\tau\lambda\hbar^{2}}\right]R(r)=0 \tag{3.105}$$

In order to writ the last equation 3.105 as a Nikiforov–Uvarov equation we have to use the following transformation

$$R\left(y\right)=y^{\upsilon}g\left(y\right) \tag{3.106}$$

Thus the equation 3.105 becomes

$$\begin{aligned}&\left(y^{2}-1\right)\left[\upsilon\left(\upsilon-1\right)y^{\upsilon-2}g\left(y\right)+2\upsilon y^{\upsilon-1}\frac{d}{dy}g\left(y\right)+y^{\upsilon}\frac{d^{2}}{dy^{2}}g\left(y\right)\right]+\\&2y\left[\upsilon y^{\upsilon-1}g\left(y\right)+y^{\upsilon}\frac{d}{dy}g\left(y\right)\right]+\frac{y^{2}}{\left(y^{2}-1\right)}\left(E_{\theta}-\frac{2\mu^{2}D_{r}}{\hbar^{2}}\right)y^{\upsilon}g\left(y\right)\\&-\frac{2\mu^{2}K}{\hbar^{2}}\frac{\left(y^{2}-1\right)}{\tau^{2}\lambda^{2}y^{2}}y^{\upsilon}g\left(y\right)+\frac{2\mu E}{\tau\lambda\hbar^{2}}y^{\upsilon}g\left(y\right)=0\end{aligned} \tag{3.107}$$

We divide by y^{υ},we get

$$\begin{aligned}&\left[\left(y^{2}-1\right)\frac{d^{2}}{dy^{2}}+\left(2\left(\upsilon+1\right)y-\frac{2\upsilon}{y}\right)\frac{d}{dy}+\frac{\left(y^{2}-1\right)}{y^{2}}\upsilon\left(\upsilon-1\right)+2\upsilon+\right.\\&\left.\frac{y^{2}}{\left(y^{2}-1\right)}\left(E_{\theta}-\frac{2\mu^{2}D_{r}}{\hbar^{2}}\right)-\frac{2\mu^{2}K}{\hbar^{2}}\frac{\left(y^{2}-1\right)}{\tau^{2}\lambda^{2}y^{2}}+\frac{2\mu E}{\tau\lambda\hbar^{2}}\right]g\left(y\right)=0\end{aligned} \tag{3.108}$$

We put $\upsilon\left(\upsilon-1\right)-\frac{2\mu^2K}{\hbar^2\tau^2\lambda^2}=0$ this require that $\upsilon=\upsilon_1=\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\tau^2\lambda^2}}$ or $\upsilon=\upsilon_2=\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\tau^2\lambda^2}}$and the equation 3.108 becomes

$$\left[\left(1-y^2\right)\frac{d^2}{dy^2}+\left(\frac{2\upsilon}{y}-2\left(\upsilon+1\right)y\right)\frac{d}{dy}+\frac{y^2}{\left(1-y^2\right)}\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)-\frac{2\mu E}{\tau\lambda\hbar^2}-2\upsilon\right]g\left(y\right)=0 \tag{3.109}$$

The accepted value of υ is the second solution because, from the expression of $R(r)$, the function $g(y)$should be nonsingular at ;$y=\pm1$

We note that the equation 3.109 possesses three singular points $y=0,\pm1$ and to reduce it to a class of known differential equation with a polynomial solution, we use a new variable

$$s=2y^2-1\Longrightarrow y=\sqrt{\frac{s+1}{2}} \tag{3.110}$$

Now we have to calculate the derivatives with respect to a new variable s

The first derivative with respect to y in terms of s is

$$\frac{d}{dy}==2\sqrt{2\left(s+1\right)}\frac{d}{ds} \tag{3.111}$$

The second derivative with respect to y in terms of s is

$$\frac{d^2}{dy^2}=4\frac{d}{ds}+8\left(s+1\right)\frac{d^2}{ds^2} \tag{3.112}$$

We use the last derivatives in equation 3.109 we get

$$\begin{gathered}\left[4\left(1-s^2\right)\frac{d^2}{ds^2}+\left(4\upsilon-2-2\left(2\upsilon+3\right)s\right)\frac{d}{ds}+\right.\\ \left.\frac{1+s}{\left(1-s\right)}\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)-\frac{2\mu E}{\tau\lambda\hbar^2}-2\upsilon\right]g\left(y\right)=0\end{gathered} \tag{3.113}$$

We put $\frac{2\mu E}{\tau\lambda\hbar^2}+2\upsilon=\varepsilon$ that give us

$$\left[4\left(1-s^2\right)\frac{d^2}{ds^2}+\left(4\upsilon-2-2\left(2\upsilon+3\right)s\right)\frac{d}{ds}+\frac{1+s}{\left(1-s\right)}\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)-\varepsilon\right]g\left(y\right)=0 \tag{3.114}$$

We divide by $4\left(1-s^2\right)$ this yield the following equation

$$\left[\frac{d^2}{ds^2}+\frac{\left(\upsilon-\frac{1}{2}\right)-\left(\upsilon+\frac{3}{2}\right)s}{\left(1-s^2\right)}\frac{d}{ds}+\frac{s^2+2s+1}{4\left(1-s^2\right)^2}\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)-\frac{\varepsilon\left(1-s^2\right)}{4\left(1-s^2\right)^2}\right]g\left(y\right)=0 \tag{3.115}$$

After some simplification the last equation.3.115 becomes

$$\left[\frac{d^2}{ds^2}+\frac{\left(\upsilon-\frac{1}{2}\right)-\left(\upsilon+\frac{3}{2}\right)s}{\left(1-s^2\right)}\frac{d}{ds}+\frac{\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}+\varepsilon\right)s^2+2\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)s+\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}-\varepsilon\right)}{4\left(1-s^2\right)^2}\right]g\left(y\right)=0 \tag{3.116}$$

To determine polynomials we compare equation .3.116 with equation 3.18 ,so

$$\sigma\left(s\right)=\left(1-s^2\right),\quad \tilde{\tau}\left(s\right)=\left(\upsilon-\frac{1}{2}\right)-\left(\upsilon+\frac{3}{2}\right)s\ \text{ and}$$
$$\tilde{\sigma}\left(s\right)=\frac{1}{4}\left[\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}+\varepsilon\right)s^2+2\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}\right)s+\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}-\varepsilon\right)\right] \tag{3.117}$$

Substituting them into equation 3.24 $\pi\left(s\right)=\left(\frac{\sigma^{\backprime}-\tilde{\tau}}{2}\right)\pm\sqrt{\left(\frac{\sigma^{\backprime}-\tilde{\tau}}{2}\right)^2-\tilde{\sigma}+\sigma k}$ we obtain

$$\pi\left(s\right)=\frac{\left(\upsilon-\frac{1}{2}\right)\left(s-1\right)}{2}\pm\frac{1}{2}\sqrt{\begin{array}{c}\left(\left(\upsilon-\frac{1}{2}\right)^2-E_\theta+\frac{2\mu^2D_r}{\hbar^2}-\varepsilon-4k\right)s^2+2\left(-E_\theta+\frac{2\mu^2D_r}{\hbar^2}-\left(\upsilon-\frac{1}{2}\right)^2\right)s\\+\left(\left(\upsilon-\frac{1}{2}\right)^2-E_\theta+\frac{2\mu^2D_r}{\hbar^2}+\varepsilon+4k\right)\end{array}} \tag{3.118}$$

The value of k is obtained from the condition that quadratic expression under the square root in 3.118 has to be completely square of first degree of polynomial therefore the discriminate of the quadratic expression under the square root that has to be zero is given as

$$\left(-E_\theta+\frac{2\mu^2D_r}{\hbar^2}-\left(\upsilon-\frac{1}{2}\right)^2\right)^2-\left(\left(\upsilon-\frac{1}{2}\right)^2-E_\theta+\frac{2\mu^2D_r}{\hbar^2}-\varepsilon-4k\right)\left(\left(\upsilon-\frac{1}{2}\right)^2-E_\theta+\frac{2\mu^2D_r}{\hbar^2}+\varepsilon+4k\right)=0 \tag{3.119}$$

And

$$\pi(s) = \frac{\left(\upsilon - \frac{1}{2}\right)(s-1)}{2} \pm$$
$$\frac{1}{2}\sqrt{\left(\left(\upsilon - \frac{1}{2}\right)^2 - E_\theta + \frac{2\mu^2 D_r}{\hbar^2} - \varepsilon - 4k\right)\left(s - \frac{\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2} + \left(\upsilon - \frac{1}{2}\right)^2\right)}{\left(\left(\upsilon - \frac{1}{2}\right)^2 - E_\theta + \frac{2\mu^2 D_r}{\hbar^2} - \varepsilon - 4k\right)}\right)} \tag{3.120}$$

The solution of the equation 3.119 give as to values for k

$$k_1 = \frac{1}{4}\left(-\varepsilon + 2\left(\upsilon - \frac{1}{2}\right)\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right) \ \textit{and}$$
$$k_2 = \frac{1}{4}\left(-\varepsilon - 2\left(\upsilon - \frac{1}{2}\right)\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right) \tag{3.121}$$

No we have to calculate $\pi(s)$ from the relation 3.120 for the two values of k

For $k_1 = \frac{1}{4}\left(-\varepsilon + 2\left(\upsilon - \frac{1}{2}\right)\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)$

$$\pi_1 = \frac{1}{2}\left[\left(2\left(\upsilon - \frac{1}{2}\right) - \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)s - \left(2\left(\upsilon - \frac{1}{2}\right) + \left(\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)\right)\right] \tag{3.122}$$

And

$$\pi_2 = \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\,(s+1) \tag{3.123}$$

For $k_2 = \frac{1}{4}\left(-\varepsilon - 2\left(\upsilon - \frac{1}{2}\right)\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)$

$$\pi_3 = \frac{1}{2}\left[\left(2\left(\upsilon - \frac{1}{2}\right) + \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)s - \left(2\left(\upsilon - \frac{1}{2}\right) - \left(\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)\right)\right] \tag{3.124}$$

And

$$\pi_4 = -\frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\,(s+1) \tag{3.125}$$

From equation 3.21 $\tau(s) = \tilde{\tau}(s) + 2\pi(s)$ and $\tilde{\tau}(s) = \left(\upsilon - \frac{1}{2}\right) - \left(\upsilon + \frac{3}{2}\right)s$

For $\pi_1$and $k_1$we have

$$\tau\left(s\right)=\left(\upsilon-\frac{5}{2}-\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)s-\left(\upsilon-\frac{1}{2}+\left(\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)\right) \tag{3.126}$$

From equation 3.23 $k=\Lambda-\pi^{\backprime}\left(s\right)\Longrightarrow\Lambda=k+\pi^{\backprime}\left(s\right)$ so

$$\Lambda=\frac{1}{4}\left(-\varepsilon+2\left(\upsilon-\frac{1}{2}\right)\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)+\frac{1}{2}\left(2\left(\upsilon-\frac{1}{2}\right)-\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right) \tag{3.127}$$

In other side we have from equation 3.22 $\Lambda_{n}+n_{r}\tau^{\backprime}+\dfrac{n_{r}\left(n_{r}-1\right)\sigma^{\backprime\backprime}}{2}=0,\quad n_{r}=0,1,2,...$thus

$$\frac{1}{4}\left(-\varepsilon+2\left(\upsilon-\frac{1}{2}\right)\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)+\frac{1}{2}\left(2\left(\upsilon-\frac{1}{2}\right)-\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)+$$
$$n_{r}\left(\upsilon-\frac{5}{2}-\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)+\frac{n\left(n-1\right)\left(-2\right)}{2}=0 \tag{3.128}$$

We use the relations $\frac{2\mu E}{\tau\lambda\hbar^{2}}+2\upsilon=\varepsilon$,$\upsilon=\upsilon_{2}=\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\mu^{2}K}{\hbar^{2}\tau^{2}\lambda^{2}}}$ and after some simplifications we find the energy as

$$\begin{aligned}E &= \frac{\hbar}{2\mu}\sqrt{\hbar^{2}\tau^{2}\lambda^{2}+8\mu^{2}K}\left(2n_{r}+1+\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)-\\ &\quad \frac{\tau\lambda\hbar^{2}}{\mu}\left(\left(2n_{r}+1\right)\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}+2n_{r}^{2}+2n_{r}+\frac{1}{2}\right)\end{aligned} \tag{3.129}$$

where $n_{r}=1,2,3,....$

Now we have to writ the expression of the radial wave functions as $R\left(s\right)=\phi\left(s\right)\rho\left(s\right)$, we first get $g(s)=\phi\left(s\right)\rho\left(s\right)$

We substitute by the expression of π_{1} and $\sigma\left(s\right)=\left(1-s^{2}\right)$ in equation 3.57 to find $\phi\left(s\right)$ as

$$\phi\left(s\right)=\frac{\left(1-s\right)^{\frac{1}{2}\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}}}{\left(1+s\right)^{\left(\upsilon-\frac{1}{2}\right)}} \tag{3.130}$$

We use the expression of $\tau\left(s\right)$ from equation 3.126 and $\sigma\left(s\right)$ to find the weight function $\rho\left(s\right)$ from equation 3.62

$$\rho\left(s\right)=Exp\left[\int\left(\frac{\left(\upsilon-\frac{1}{2}-\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)s-\left(\upsilon-\frac{1}{2}+\left(\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)\right)}{\left(1-s^{2}\right)}\right)ds\right] \tag{3.131}$$

After the calculation of the integral we find

$$\rho(s) = \frac{(1-s)^{\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}}}{(1+s)^{\left(\upsilon - \frac{1}{2}\right)}} \tag{3.132}$$

the $y_n(s)$part is given by Rodrigues relation

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} \left[(\sigma(s))^n \rho(s) \right] \tag{3.133}$$

Where$\rho(s) = \frac{(1-s)^{\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}}}{(1+s)^{\left(\upsilon - \frac{1}{2}\right)}}$. and $\sigma(s) = (1-s^2)$ equation 3.133 stands for the Romanovski polynomials as

$$y_n(s) = p_n^{\left(\frac{1}{2} - \upsilon, \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)}(s) = \frac{C_n (1+s)^{\left(\upsilon - \frac{1}{2}\right)}}{(1-s)^{\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}}} \frac{d^n}{ds^n} \left[(1-s)^{n + \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}} (1+s)^{n - \left(\upsilon - \frac{1}{2}\right)} \right] \tag{3.134}$$

From equation 3.130 and 3.134 the function $g(s)$ is

$$g(s) = C_n \frac{(1-s)^{\frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}}}{(1+s)^{\left(\upsilon - \frac{1}{2}\right)}} p_n^{\left(\frac{1}{2} - \upsilon, \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)}(s) \tag{3.135}$$

Hence, $R(s)$ can be written in the following form $R(y) = y^\upsilon g(y), s = 2y^2 - 1$

$$R(y) = C_n \frac{(2-2y^2)^{\frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}}}{(2y^2)^{\left(\upsilon - \frac{1}{2}\right)}} p_n^{\left(\frac{1}{2} - \upsilon, \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)} \left(2y^2 - 1\right) y^\upsilon \tag{3.136}$$

We have $\upsilon = \upsilon_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8\mu^2 K}{\hbar^2 \tau^2 \lambda^2}}$ and $y = \sqrt{1 + \tau\lambda r^2}$ so the radial wave function can be written as

$$\begin{aligned} R(r) &= C_n \left(2\tau\lambda r^2\right)^{\frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}} p_n^{\left(-\frac{1}{2}\sqrt{1 + \frac{8\mu^2 K}{\hbar^2\lambda^2}}, \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)} \\ &\quad \left(2\tau\lambda r^2 + 1\right) \sqrt{1 + \tau\lambda r^2}^{\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{8\mu^2 K}{\hbar^2\lambda^2}}} (2)^{\left(-\frac{1}{2}\sqrt{1 + \frac{8\mu^2 K}{\hbar^2\lambda^2}}\right)} \end{aligned} \tag{3.137}$$

C_n is the normalization constant

Solution of the Radial Equation in Anti- deSitter Space ($\tau = -1$) By the same way of deSitter case we find the energy and wave function of anti deSitter space

The deformed energy is

$$E=\frac{\hbar}{2\mu}\sqrt{\hbar^2\lambda^2+8\mu K}\left(2n_r+1+\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)+$$
$$\frac{\lambda\hbar^2}{\mu}\left((2n_r+1)\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}+2n_r^2+2n_r+\frac{1}{2}\right) \tag{3.138}$$

The radial wave function is

$$R\left(r\right)=C_n\left(-2\lambda r^2\right)^{\frac{1}{2}\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}}p_n^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}},\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)}$$
$$\left(-2\lambda r^2+1\right)\sqrt{1+\tau\lambda r^2}^{\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}}}(2)^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}}\right)} \tag{3.139}$$

Energy and Wave Function

de Sitter Space We substitute the constant of separation from equation 1.34 in the energy expression 3.129 ,we get the deformed energy as follows:

$$E=\frac{\hbar}{2\mu}\sqrt{\hbar^2\lambda^2+8\mu^2K}\left(2n_r+1+\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)+\frac{2\mu^2D_r}{\hbar^2}}\right)-$$
$$\frac{\lambda\hbar^2}{\mu}\left((2n_r+1)\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)+\frac{2\mu^2D_r}{\hbar^2}}+2n_r^2+2n_r+\frac{1}{2}\right) \tag{3.140}$$

We deduce the wave function of our system $\psi\left(r,\theta\right)=\psi(r,\theta)=R(r)\Theta(\theta)$ from the angular part 1.35 and radial part 3.137

$$\psi\left(r,\theta\right)=N\left(2\lambda r^2\right)^{\frac{1}{2}\sqrt{\frac{1}{4}c_{2m}(2\alpha)+\frac{2\mu^2D_r}{\hbar^2}}}p_n^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}},\sqrt{\frac{1}{4}c_{2m}(2\alpha)+\frac{2\mu^2D_r}{\hbar^2}}\right)}$$
$$\left(2\lambda r^2+1\right)\sqrt{1+\lambda r^2}^{\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}}}(2)^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}}\right)}\Theta(\theta) \tag{3.141}$$

$n_r=1,2,3,....,m=1,2,3,....$

$c_{2m}(p)$ Mathieu characteristic values and $\Theta(\theta)$ is Mathieufunction

Anti de Sitter Space We substitute the constant of separation from equation 1.34 in the energy expression 3.138 ,we get the deformed energy as follows:

$$E=\frac{\hbar}{2\mu}\sqrt{\hbar^{2}\lambda^{2}+8\mu^{2}K}\left(2n_{r}+1+\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)+$$
$$\frac{\lambda\hbar^{2}}{\mu}\left(\left(2n_{r}+1\right)\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}+2n_{r}^{2}+2n_{r}+\frac{1}{2}\right) \tag{3.142}$$

$n_r = 1, 2, 3, ..., m = 1, 2, 3, ...$

$$\psi\left(r,\theta\right)=N\left(-2\lambda r^{2}\right)^{\frac{1}{2}\sqrt{\frac{1}{4}c_{2m}(2\alpha)+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}}p_{n}^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^{2}K}{\hbar^{2}\lambda^{2}}},\sqrt{\frac{1}{4}c_{2m}(2\alpha)+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)}$$
$$\left(-2\lambda r^{2}+1\right)\sqrt{1-\lambda r^{2}}^{\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{8\mu^{2}K}{\hbar^{2}\lambda^{2}}}}\left(2\right)^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^{2}K}{\hbar^{2}\lambda^{2}}}\right)}\Theta(\theta) \tag{3.143}$$

$c_{2m}(p)$ Mathieu characteristic values and $\Theta(\theta)$ is Mathieufunction

For the potential The potential $\mathbf{V}_4\left(r,\theta\right)=\boldsymbol{\mu}\left[kr^{2}+\frac{1}{r^{2}}\left(\alpha\cos\theta\right)\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_3\left(r,\theta\right)$ when we put $D_r \longrightarrow 0$ so

de Sitter Space: The deformed energy is

$$E=\frac{\hbar}{2\mu}\sqrt{\hbar^{2}\lambda^{2}+8\mu^{2}K}\left(2n_{r}+1+\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)}\right)-$$
$$\frac{\lambda\hbar^{2}}{\mu}\left(\left(2n_{r}+1\right)\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)}+2n^{2}+2n+\frac{1}{2}\right) \tag{3.144}$$

The deformed wave function is

$$\psi\left(r,\theta\right)=N\left(2\lambda r^{2}\right)^{\frac{1}{2}\sqrt{\frac{1}{4}c_{2m}(2\alpha)}}p_{n}^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^{2}K}{\hbar^{2}\lambda^{2}}},\sqrt{\frac{1}{4}c_{2m}(2\alpha)}\right)}$$
$$\left(2\lambda r^{2}+1\right)\sqrt{1+\lambda r^{2}}^{\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{8\mu^{2}K}{\hbar^{2}\lambda^{2}}}}\left(2\right)^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^{2}K}{\hbar^{2}\lambda^{2}}}\right)}\Theta(\theta) \tag{3.145}$$

$n_r = 1, 2, 3, ..., m = 1, 2, 3, ...$

$c_{2m}(p)$ Mathieu characteristic values and $\Theta(\theta)$ is Mathieufunction

Anti de Sitter Space: The deformed energy is

$$E = \frac{\hbar}{2\mu}\sqrt{\hbar^2\lambda^2 + 8\mu^2 K}\left(2n_r + 1 + \sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)}\right) +$$
$$\frac{\lambda\hbar^2}{\mu}\left(\left(2n_r + 1\right)\sqrt{\frac{1}{4}c_{2m}\left(2\alpha\right)} + 2n_r^2 + 2n_r + \frac{1}{2}\right) \tag{3.146}$$

The deformed wave function is

$$\psi\left(r,\theta\right) = N\left(-2\lambda r^2\right)^{\frac{1}{2}\sqrt{\frac{1}{4}c_{2m}(2\alpha)}} p_n^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\lambda^2}},\sqrt{\frac{1}{4}c_{2m}(2\alpha)}\right)}$$
$$\left(-2\lambda r^2 + 1\right)\sqrt{1-\lambda r^2}^{\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\lambda^2}}}\left(2\right)^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\lambda^2}}\right)}\Theta(\theta) \tag{3.147}$$

$n_r = 1, 2, 3, ..., m = 1, 2, 3, ...$

$c_{2m}(p)$ Mathieu characteristic values and $\Theta(\theta)$ is Mathieufunction

We summarize the previous results in $(Tables\ 3.5\ and\ 3.6)$

$\delta_1 = \sqrt{\frac{1}{4} + \left(E_\theta + \frac{2\mu^2 D_r}{\hbar^2}\right) + n_r(n_r+1) - (2n_r+1)\sqrt{\left(E_\theta + \frac{2\mu^2 D_r}{\hbar^2}\right)}}$,and $\eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$

$\varepsilon = \sqrt{E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}, \varsigma = -\frac{1}{2}\sqrt{1 + \frac{8\mu^2 K}{\hbar^2\lambda^2}}$

E_θ and $\Theta(\theta)$ are shown in$(Tabels\ 1.2\ and\ 1.3)$

3.4 Discussion

We remark that the expression of energies contains the ordinary energy term and an additional correction term proportional to the deformation parameter λ, . It should be noted here that

for the potentials which contain the Colombian potential the first term of the correction is proportional to n_r^2 and so it is equivalent to the energy of a nonrelativistic quantum particle moving in a square well potential. In our case, the boundaries of the well are placed at $\pm\frac{\pi}{2}\sqrt{\lambda}$. The second term in the correction contains the number m. We also notice that in the deSitter case the deformed energy increase comparing to the ordinary energy unlike the anti deSitter when the energy is decrease and is inversely proportional to the deformation parameter λ,

For the potentials which contain oscillator potential unlike the Colombian case ,.in the deSitter case the deformed energy decrease comparing to the ordinary energy unlike the anti deSitter when the energy is increase and it appears that the momentum of the oscillator is affected by the deformation

$V(r,\theta)$	$Space$	E
$\mu\left[-\frac{H}{r}+\frac{f(\theta)}{r^2}\right]$	dS	$-2\frac{\mu^3H^2}{\hbar^2}\left(2n_r+2\sqrt{-E_\theta+\frac{2\mu D_r}{\hbar^2}}+1\right)^{-2}$ $-\frac{\lambda\hbar^2}{8\mu}\left[(2n_r+1)\left(2n_r+1+4\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)-1\right]$
$\mu\left[-\frac{H}{r}+\frac{f(\theta)}{r^2}\right]$	Ads	$-2\frac{\mu^3H^2}{\hbar^2}\left(2n_r+2\sqrt{-E_\theta+\frac{2\mu D_r}{\hbar^2}}+1\right)^{-2}$ $+\frac{\lambda\hbar^2}{8\mu}\left[(2n_r+1)\left(2n_r+1+4\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)-1\right]$
$\mu\left[kr^2+\frac{f(\theta)}{r^2}\right]$	dS	$\frac{\hbar}{2\mu}\sqrt{\hbar^2\lambda^2+8\mu K}\left(2n_r+1+\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)$ $-\frac{\lambda\hbar^2}{\mu}\left((2n_r+1)\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}+2n_r^2+2n_r+\frac{1}{2}\right)$
$\mu\left[kr^2+\frac{f(\theta)}{r^2}\right]$	AdS	$\frac{\hbar}{2\mu}\sqrt{\hbar^2\lambda^2+8\mu K}\left(2n_r+1+\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)$ $+\frac{\lambda\hbar^2}{\mu}\left((2n_r+1)\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}+2n_r^2+2n_r+\frac{1}{2}\right)$

Table 3.5: The expression of deformed energy in 2D space

$V(r,\theta)$	$Space$	ψ
$\mu\left[-\frac{H}{r}+\frac{f(\theta)}{r^2}\right]$	dS	$N\left(1-\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)}\left(1+\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)}\times$ $P_n^{\left(-\delta_1+\frac{\eta}{2\delta_1},-\delta_1-\frac{\eta}{2\delta_1}\right)}\left(\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)\times\Theta(\theta)$
$\mu\left[-\frac{H}{r}+\frac{f(\theta)}{r^2}\right]$	Ads	$N\left(1+\frac{1-\lambda r^2}{\lambda r}\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_1\right)}e^{\frac{\eta}{2\delta_1}\tan^{-1}(s)}R_n^{\left(-\delta_1,\frac{\eta}{\delta_1}\right)}\left(\frac{\sqrt{1-\lambda r^2}}{\sqrt{\lambda}r}\right)\times$ $\Theta(\theta)$
$\mu\left[kr^2+\frac{f(\theta)}{r^2}\right]$	dS	$N\left(2\lambda r^2\right)^{\frac{1}{2}\varepsilon}p_n^{(\varsigma,\varepsilon)}\left(2\lambda r^2+1\right)\sqrt{1+\lambda r^2}^{\frac{1}{2}+\varsigma}(2)^{(\varsigma)}\Theta(\theta)$
$\mu\left[kr^2+\frac{f(\theta)}{r^2}\right]$	AdS	$N\left(-2\lambda r^2\right)^{\frac{1}{2}\varepsilon}p_n^{(\varsigma,\varepsilon)}\left(1-2\lambda r^2\right)\sqrt{1-\lambda r^2}^{\frac{1}{2}+\varsigma}(2)^{(\varsigma)}\Theta(\theta)$

Table 3.6: The expression of deformed wave function in 2D space

Chapter 4

Studies of N-C Potentials in 3D (dS and AdS) Spaces

4.1 3D Schrödinger Equation of N-C Potentials in Deformed Space

We consider the following 3D stationary Schrödinger equation with a non-central potential

$$\left[\frac{\mathbf{p}^2}{2\mu}+\mu\left(V(r)+\frac{f(\theta)}{\boldsymbol{r}^2}\right)\right]\psi(r,\theta)=E\psi(r,\theta) \tag{4.1}$$

In order to include the effect of EUP on the above Schrödinger equation, we use the transformations 3.6a and 3.6b to obtain:

$$\left[\frac{1}{2\mu}\left(\left(1+\tau\lambda r^2\right)p^2+\tau\lambda rp\right)+\mu\left(V(\frac{r}{\sqrt{1+\tau\lambda r^2}})+\frac{\left(1+\tau\lambda r^2\right)f(\theta)}{r^2}\right)\right]\psi(r,\theta)=E\psi(r,\theta) \tag{4.2}$$

We use the spheric coordinates

$$\begin{gathered}\left[\left(1+\tau\lambda r^2\right)\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)+\tau\lambda r\frac{\partial}{\partial r}-\right.\\ \left.\frac{2\mu^2}{\hbar^2}\left(V(\frac{r}{\sqrt{1+\tau\lambda r^2}})+\frac{\left(1+\tau\lambda r^2\right)f(\theta)}{r^2}\right)\right]\psi=-\frac{2\mu}{\hbar^2}E\psi\end{gathered} \tag{4.3}$$

After some simplification we get

$$\begin{gathered}\left[\left(1+\tau\lambda r^2\right)\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\tau\lambda r\frac{\partial}{\partial r}+\frac{2\mu}{\hbar^2}E-\frac{2\mu^2}{\hbar^2}V(\frac{r}{\sqrt{1+\tau\lambda r^2}})+\right.\\ \left.\frac{\left(1+\tau\lambda r^2\right)}{r^2}\left[\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-\frac{2\mu^2}{\hbar^2}f(\theta)\right]\right]\psi=0\end{gathered} \tag{4.4}$$

In order to separate the variables, we write the solution as $\psi(r,\theta)=r^{-1/2}R(r)e^{im\varphi}\Theta(\theta)$ and this enables us to split the equation into two parts, one angular and the other radial

The first derivative $\frac{\partial\psi}{\partial r}$ in terms of a new function is

$$\frac{\partial\psi}{\partial r}=\frac{\partial}{\partial r}r^{-1/2}R(r)e^{im\varphi}\Theta(\theta)=-\frac{1}{2}r^{-\frac{3}{2}}R(r)e^{im\varphi}\Theta(\theta)+r^{-1/2}\frac{\partial}{\partial r}R(r)e^{im\varphi}\Theta(\theta) \tag{4.5}$$

The second derivative $\frac{\partial^2\psi}{\partial r^2}$ in terms of a new function is

$$\frac{\partial^2\psi}{\partial r^2}=\frac{3}{4}r^{-\frac{5}{2}}R(r)e^{im\varphi}\Theta(\theta)-r^{-\frac{3}{2}}\frac{\partial}{\partial r}R(r)e^{im\varphi}\Theta(\theta)+r^{-1/2}\frac{\partial^2}{\partial r^2}R(r)e^{im\varphi}\Theta(\theta) \tag{4.6}$$

We substitute the derivatives in the Schrödinger equation 4.4

$$\begin{gathered}\Bigg[\left(\sqrt{1+\tau\lambda r^2}\frac{\partial}{\partial r}\right)^2+\frac{(1+\tau\lambda r^2)}{r}\frac{\partial}{\partial r}+\\ -\frac{1}{4}\frac{(1+\tau\lambda r^2)}{r^2}-\frac{1}{2}\tau\lambda+\frac{2\mu}{\hbar^2}E-\frac{2\mu^2}{\hbar^2}V(\frac{r}{\sqrt{1+\tau\lambda r^2}})+\\ \frac{(1+\tau\lambda r^2)}{r^2}\left[\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-\frac{2\mu^2}{\hbar^2}f(\theta)\right]\Bigg]R(r)e^{im\varphi}\Theta(\theta)=0\end{gathered} \tag{4.7}$$

The last equation can be written as two equations the angular equation and the radial equation as

$$\left[\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}-\frac{m^2}{\sin^2\theta}-\frac{2\mu^2}{\hbar^2}f(\theta)\right]\Theta(\theta)=E_\theta\Theta(\theta) \tag{4.8}$$

$$\begin{gathered}\left[\left(\sqrt{1+\tau\lambda r^2}\frac{d}{dr}\right)^2+\frac{(1+\tau\lambda r^2)}{r}\frac{d}{dr}-\frac{\left(-E_\theta+\frac{1}{4}\right)(1+\tau\lambda r^2)}{r^2}-\frac{2\mu^2}{\hbar^2}V(\frac{r}{\sqrt{1+\tau\lambda r^2}})\right]R(r)=\\ -\left(\frac{2\mu E}{\hbar^2}+\frac{\tau\lambda}{2}\right)R(r)\end{gathered} \tag{4.9}$$

4.2 Non-Relativistic Solutions of N-C Potentials in 3D Deformed Space

Case1 $V_1(r,\theta)=\mu\left[-\frac{H}{r}+\frac{D_r}{r^2}+\frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta+\beta\cos\theta+\gamma)\sin^{-2}\theta\right]$

Solution of Angular Equation We note that the angular equation 4.8 is same to equation 2.14 of chapter 2 and his solutions are appear in equations 2.36 195 and 214 depend to $f(\theta)$

Solution of Radial Equation For this potential the radial equation 4.9 becomes

$$\left[\left(\sqrt{1+\tau\lambda r^2}\frac{d}{dr}\right)^2+\frac{\left(1+\tau\lambda r^2\right)}{r}\frac{d}{dr}-\frac{\left(-E_\theta+\frac{2\mu^2}{\hbar^2}D_r+\frac{1}{4}\right)\left(1+\tau\lambda r^2\right)}{r^2}+\frac{2\mu^2}{\hbar^2}\frac{H\sqrt{1+\tau\lambda r^2}}{r}\right]R(r)$$
$$=-\left(\frac{2\mu E}{\hbar^2}+\frac{\tau\lambda}{2}\right)R(r) \quad (4.10)$$

In order to solve this equation , we use the following transformations:

$$s=\frac{\sqrt{1+\tau\lambda r^2}}{\sqrt{\lambda}r} \quad (4.11)$$

Then, the new form of 4.10 becomes:

$$\left[\left(1-\tau s^2\right)^2\frac{d^2}{ds^2}-\tau s\left(1-\tau s^2\right)\frac{d}{ds}-\left(-E_\theta+\frac{2\mu^2}{\hbar^2}D_r+\frac{1}{4}\right)s^2+\eta s+\varepsilon\right]R_{1,2}(s)=0 \quad (4.12)$$

where

$$\eta=\frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}} \text{ and } \varepsilon=\frac{2\mu E}{\hbar^2}+\frac{\tau\lambda}{2} \quad (4.13)$$

We divide the last equation by $\left(1-\tau s^2\right)^2$,that give arise

$$\left[\frac{d^2}{ds^2}-\frac{\tau s}{\left(1-\tau s^2\right)}\frac{d}{ds}+\frac{1}{\left(1-\tau s^2\right)^2}\left(-\left(-E_\theta+\frac{2\mu^2}{\hbar^2}D_r+\frac{1}{4}\right)s^2+\eta s+\varepsilon\right)\right]R_{1,2}(s)=0 \quad (4.14)$$

de Sitter Space($\tau=1$) This case is represented by the equation 4.14 with ($\tau=1$) as

$$\left[\frac{d^2}{ds^2}-\frac{s}{\left(1-s^2\right)}\frac{d}{ds}+\frac{1}{\left(1-s^2\right)^2}\left(-\left(-E_\theta+\frac{2\mu^2}{\hbar^2}D_r+\frac{1}{4}\right)s^2+\eta s+\varepsilon\right)\right]R_{1,2}(s)=0 \quad (4.15)$$

To determine polynomials we compare equation 4.15 with equation. 3.18,so

$$\sigma(s)=\left(1-s^2\right), \quad \tilde{\tau}(s)=-s \text{ and } \tilde{\sigma}(s)=-\left(-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}+\frac{1}{4}\right)s^2+\eta s+\varepsilon \quad (4.16)$$

Substituting them into Equation. 3.24: $\pi(s)=\left(\frac{\sigma'-\tilde{\tau}}{2}\right)\pm\sqrt{\left(\frac{\sigma'-\tilde{\tau}}{2}\right)^2-\tilde{\sigma}+\sigma k}$ we obtain

$$\pi(s)=\frac{-s}{2}\pm\sqrt{\left(\frac{1}{4}+\left(-E_\theta^{(2m)}+\frac{2\mu^2 D_r}{\hbar^2}+\frac{1}{4}\right)-k\right)s^2-\eta s+k-\varepsilon} \quad (4.17)$$

The value of k is obtained from the condition that quadratic expression under the square root in 4.17 has to be completely square of first degree of polynomial

$$\left(\frac{1}{2}-\left(-E_{\theta}^{(2m)}+\frac{2\mu^2 D_r}{\hbar^2}\right)-k\right)s^2+\eta s+k-\varepsilon=\left(\frac{1}{2}-\left(-E_{\theta}^{(2m)}+\frac{2\mu^2 D_r}{\hbar^2}\right)-k\right)(s-s_0)^2 \tag{4.18}$$

And

$$\pi(s)=\frac{-s}{2}\pm\sqrt{\left(\frac{1}{2}-\left(-E_{\theta}^{(2m)}+\frac{2\mu^2 D_r}{\hbar^2}\right)-k\right)}(s-s_0) \tag{4.19}$$

The solution of equation 4.18 obtains the following possible solutions for each k:

$$\pi(s)=\begin{cases}\pi_{1,2}=\left(\frac{-1}{2}\pm\delta_1\right)s\mp\frac{\eta}{2\delta_1}\text{ for }k_1=\frac{1}{2}\left[\varepsilon+\frac{1}{2}-\left(E_{\theta}^{(2m)}-\frac{2\mu^2 D_r}{\hbar^2}\right)+\sqrt{\Delta}\right]\\ \pi_{3,4}=\left(\frac{-1}{2}\pm\delta_2\right)s\mp\frac{\eta}{2\delta_2}\text{ for }k_2=\frac{1}{2}\left[\varepsilon+\frac{1}{2}-\left(E_{\theta}^{(2m)}-\frac{2\mu^2 D_r}{\hbar^2}\right)-\sqrt{\Delta}\right]\end{cases} \tag{4.20}$$

With

$$\delta_{1,2}=\sqrt{\frac{1}{2}-\left(E_{\theta}^{(2m)}-\frac{2\mu^2 D_r}{\hbar^2}\right)-k_{1,2}}\text{ and }\Delta=\left(\varepsilon-\frac{1}{2}+\left(E_{\theta}^{(2m)}-\frac{2\mu^2 D_r}{\hbar^2}\right)\right)^2-\eta^2 \tag{4.21}$$

Here, we choose $k_1$and $\pi_1$because they give as the limit of the ordinary space so that yield

$$\tau(s)=2(\delta_1-1)s-\frac{\eta}{\delta_1} \tag{4.22}$$

And

$$k=\Lambda-\pi(s) \tag{4.23}$$

From equation 3.22, and the expressions of $\tau(s)$ and $\sigma(s)=(1-s^2)$ we calculate:

$$\Lambda=k_1-\frac{1}{2}+\delta_1=n_r(n_r+1-2\delta_1)\implies k_1=\frac{1}{2}-\delta_1(2n_r+1)+n_r(n_r+1),\ n_r=0,1,2,... \tag{4.24}$$

By the same method of 2D space of chapter 2 , the energy eigenvalues are found as:

$$k_1-\frac{1}{2}+\delta_1=n_r(n_r+1-2\delta_1)\implies k_1=\frac{1}{2}-\delta_1(2n_r+1)+n_r(n_r+1) \tag{4.25}$$

Now we have substitute the expression of k_1 and $\delta_1$from equation 3.44 and equation 3.45 to find the energy

$k_1=\frac{1}{2}\left[\varepsilon+\frac{1}{4}-\left(E_{\theta}^{(2m)}-\frac{2\mu^2 D_r}{\hbar^2}\right)+\sqrt{\left(\varepsilon-\frac{1}{4}+\left(E_{\theta}^{(2m)}-\frac{2\mu^2 D_r}{\hbar^2}\right)\right)^2-\eta^2}\right]$ and

$\delta_1=\sqrt{\frac{1}{4}-\left(E_{\theta}^{(2m)}-\frac{2\mu^2 D_r}{\hbar^2}\right)-k_1}$

The energy is

$$E_{n,l} = -\frac{\mu^3 H^2}{2\hbar^2}\left(n_r + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu^2}{\hbar^2}D_r - E_\theta}\right)^{-2} - \frac{\lambda\hbar^2}{2m}\left(n_r^2 + E_\theta - \frac{2\mu^2}{\hbar^2}D_r - 1\right) \quad (4.26)$$

where $n_r = 1, 2, 3, ...$

Now let us find the corresponding eigenfunctions. Taking the expression of $\pi_1(s)$ from 4.20, the $\phi(s)$ part is defined by the same way of previous sections as

$$\phi(s) = (1+s)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)}(1-s)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)} \quad (4.27)$$

and according to the form of $\sigma(s) = (1-s^2)$, the $y(s)$ part is given by Rodrigues relation:

$$y_n(s) = \frac{C_n}{\rho(s)}\frac{d^n}{ds^n}\left[\left(1-s^2\right)^n \rho(s)\right] \quad (4.28)$$

where $\rho(s) = (1+s)^{\left(-\delta_1-\frac{\eta}{2\delta_1}\right)}(1-s)^{-\left(\delta_1-\frac{\eta}{2\delta_1}\right)}$. The expression of $y_n(s)$ stands for the Jacobi polynomials as:

$$y_n(s) \equiv P_{n_r}^{\left(-\delta_1-\frac{\eta}{2\delta_1}, -\delta_1+\frac{\eta}{2\delta_1}\right)}(s) \quad (4.29)$$

Hence, $R(s)$ can be written in the following form:

$$R(s) = C_n(1-s)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)}(1+s)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)}P_{n_r}^{\left(-\delta_1-\frac{\eta}{2\delta_1}, -\delta+\frac{\eta}{2\delta_1}\right)}(s) \quad (4.30)$$

In terms of the variables r, θ and φ, we can now write $R(r)$ as follows

$$\begin{aligned} R(r) = C_n &\left(1-\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)}\left(1+\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)} \\ &P_{n_r}^{\left(-\delta_1-\frac{\eta}{2\delta_1}, -\delta+\frac{\eta}{2\delta_1}\right)}\left(\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right) \end{aligned} \quad (4.31)$$

where C_n is a normalization constant,$\delta_1 = \sqrt{\frac{1}{2} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - k_1}, \eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$and

$k_1 = \frac{1}{2}\left[\varepsilon + \frac{1}{2} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) + \sqrt{\left(\varepsilon - \frac{1}{2} + \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right)\right)^2 - \eta^2}\right]$

Anti- deSitter Space($\tau = -1$) The radial equation of the anti-deSitter space is

$$\left[\frac{d^2}{ds^2}-\frac{s}{(1+s^2)}\frac{d}{ds}+\frac{1}{(1+s^2)^2}\left(-\left(-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}D_r+\frac{1}{4}\right)s^2+\eta s+\varepsilon\right)\right]R_{1,2}(s)=0 \tag{4.32}$$

To determine polynomials we compare Equation. 4.32 with Equation. 3.18,so we get

$$\sigma(s)=\left(1+s^2\right)\ ,\ \tilde{\tau}(s)=s \text{ and } \tilde{\sigma}(s)=-\left(-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}D_r+\frac{1}{4}\right)s^2+\eta s+\varepsilon \tag{4.33}$$

Substituting them into 3.24, we obtain:

$$\pi(s)=\frac{s}{2}\pm\sqrt{\left(k+\frac{1}{4}+\left(-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}D_r+\frac{1}{4}\right)\right)s^2-\eta s+k-\varepsilon} \tag{4.34}$$

The constant k is determined in the same way as in deSitter case. Therefore, we get:

$$\pi(s)=\begin{cases}\pi_{1,2}=\left(\frac{1}{2}\pm\delta_1\right)s\mp\frac{\eta}{2\delta_1^{\backprime}} \text{ for } k_1'=\frac{1}{2}\left[\varepsilon-\frac{1}{2}+E_\theta-\frac{2\mu^2 D_r}{\hbar^2}D_r-\sqrt{\triangle}\right]\\ \pi_{3,4}=\left(\frac{1}{2}\pm\delta_2\right)s\mp\frac{\eta}{2\delta_2^{\backprime}} \text{ for } k_2'=\frac{1}{2}\left[\varepsilon-\frac{1}{2}+E_\theta-\frac{2\mu^2 D_r}{\hbar^2}D_r+\sqrt{\triangle}\right]\end{cases} \tag{4.35}$$

where:

$$\delta_{1,2}'=\sqrt{\frac{1}{2}-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}D_r+k_{1,2}'} \text{ and } \Delta=\left(\varepsilon+\frac{1}{2}-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}D_r\right)^2+\eta^2 \tag{4.36}$$

Here, we choose$k_2^{\backprime}$ and π_4 for the limits in ordinary space so that we have:

$$\tau(s)=2\left(1-\delta_2'\right)s-\frac{\eta}{\delta_2^{\backprime}} \tag{4.37}$$

And

$$k=\Lambda-\pi(s) \tag{4.38}$$

From equation 3.22, and the expressions of $\tau(s)$ and $\sigma(s)=(1+s^2)$ we calculate:

$$\Lambda=k_2^{\backprime}+\frac{1}{2}-\sqrt{\frac{1}{2}+E_\theta+\frac{2\mu^2 D_r}{\hbar^2}D_r+k_2^{\backprime}}=-n_r\left(n_r+1-2\sqrt{\frac{1}{2}+E_\theta+\frac{2\mu^2 D_r}{\hbar^2}D_r+k_2^{\backprime}}\right) \tag{4.39}$$

Hence, the energy eigenvalues are found as:

$$E_{n,l,m}=-\frac{\mu^3 H^2}{2\hbar^2}\left(n_r+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu^2}{\hbar^2}D_r-E_\theta}\right)^{-2}+\frac{\lambda\hbar^2}{2m}\left(n_r^2+E_\theta-\frac{2\mu^2}{\hbar^2}D_r-1\right) \tag{4.40}$$

Now, to deduce the complete expression of the wave functions ψ_n, we use the expression

4.35 of $\pi_2(s)$ as follows:

$$\phi(s) = \left(1+s^2\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_2^{\backslash}\right)} e^{\frac{-\eta}{2\delta_2^{\backslash}}\tan^{-1}(s)} \tag{4.41}$$

and according to the form of $\sigma(s) = (1+s^2)$, the $y(s)$ part is given by Rodrigues relation:

$$y_n(s) = \frac{C_n^{\backslash}}{\rho(s)} \frac{d^n}{ds^n}\left[\left(1+s^2\right)^n \rho(s)\right] \tag{4.42}$$

where $\rho(s) = (1+s^2)^{-\delta_1^{\backslash}} e^{\frac{\eta}{\delta_2^{\backslash}}\tan^{-1}(s)}$.The function $y_n(s)$stands for the Romanovski polynomials [116] as:

$$y_n(s) \equiv R_n^{\left(-\delta_2^{\backslash}, \frac{-\eta}{\delta_2^{\backslash}}\right)}(s) = \frac{C_n^{\backslash}}{(1+s^2)^{-\delta_2^{\backslash}} e^{\frac{-\eta}{\delta_2^{\backslash}}\tan^{-1}(s)}} \frac{d^n}{ds^n}\left[\left(1+s^2\right)^{n-\delta_1^{\backslash}} e^{\frac{-\eta}{\delta_2^{\backslash}}\tan^{-1}(s)}\right] \tag{4.43}$$

Consequently, the expression of $R(s)$ is written as:

$$R(s) = C_n\left(1+s^2\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_2'\right)} e^{\frac{-\eta}{2\delta_1}\tan^{-1}(s)} R_n^{\left(-\delta_1', \frac{-\eta}{\delta_1}\right)}(s) \tag{4.44}$$

In terms of the variables r, θ and φ, we can now write $R(r)$ as follows

$$R(s) = C_n\left(1+\frac{1-\lambda r^2}{\lambda r}\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_2'\right)} e^{\frac{-\eta}{2\delta_1}\tan^{-1}(s)} R_n^{\left(-\delta_2', \frac{-\eta}{\delta_2}\right)}\left(\frac{\sqrt{1-\lambda r^2}}{\sqrt{\lambda} r}\right) \tag{4.45}$$

where C_n is a normalization constant,$\delta_2' = \sqrt{\frac{1}{2} - E_\theta + \frac{2\mu^2 D_r}{\hbar^2} D_r + k_2'}$, $\eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$and $k_2' = \frac{1}{2}\left[\varepsilon - \frac{1}{2} + E_\theta - \frac{2\mu^2 D_r}{\hbar^2} D_r + \sqrt{\left(\varepsilon + \frac{1}{2} - E_\theta + \frac{2\mu^2 D_r}{\hbar^2} D_r\right)^2 + \eta^2}\right]$ with C_n is a normalization constant.

Energy and Wave Function

deSitter Space We substitute the constant of separation 2.36 the expression of energy 4.26 ,we find the final expression of energy as

$$E_{n_r,l,m} = -\frac{\mu^3 H^2}{2\hbar^2}\left[n_r + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu^2}{\hbar^2} D_r - \alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2}\right]^{-2} - \frac{\lambda\hbar^2}{2m}\left(n_r^2 + \alpha - \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2 - \frac{2\mu^2}{\hbar^2} D_r - 1\right) \tag{4.46}$$

$n_r = 0, 1, 2, ..., l = 0, 1, 2, ...$and $m = 0, \pm 1, \pm 2, ...$

We deduce the wave function of our system $\psi(r, \theta, \varphi) = \exp(im\varphi) R(r)\Theta(\theta)$ from the angular part 2.35 and radial part 4.31

$$\psi_1 = N \exp(im\varphi) \left(1 - \frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda} r}\right)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)} \left(1 + \frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda} r}\right)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)}$$
$$P_{n_r}^{\left(-\delta_1-\frac{\eta}{2\delta_1}, -\delta+\frac{\eta}{2\delta_1}\right)} \left(\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda} r}\right) \cos^{2\rho}\left(\frac{\theta}{2}\right) \left(1 - \cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma} \times$$
$$F(-l, l+1+(m^2+\alpha-\beta+\gamma)^{1/2} + (m^2+\alpha+\beta+\gamma)^{1/2}; 1+(m^2+\alpha-\beta+\gamma)^{1/2}; \cos^2\left(\frac{\theta}{2}\right)) \tag{4.47}$$

Where $\delta_1 = \sqrt{\frac{1}{2} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - k_1}, \eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$
$\rho = \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}$,$\sigma = \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}$

Anti deSitter space We substitute the constant of separation 2.36 the expression of energy 4.40 ,we find the final expression of energy as

$$E_{n_r,l,m} = -\frac{\mu^3 H^2}{2\hbar^2}\left[n_r + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu^2}{\hbar^2} D_r - \alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2}\right]^{-2} +$$
$$\frac{\lambda\hbar^2}{2m}\left(n_r^2 + \alpha - \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2 - \frac{2\mu^2}{\hbar^2} D_r - 1\right) \tag{4.48}$$

$n_r = 0, 1, 2, ..., l = 0, 1, 2, ...$and $m = 0, \pm 1, \pm 2, ...$

We deduce the wave function of our system $\psi(r, \theta, \varphi) = \exp(im\varphi) R(r)\Theta(\theta)$ from the angular part 2.35 and radial part 4.45

$$\psi_1 = N \exp(im\varphi) \left(1 + \frac{1-\lambda r^2}{\lambda r}\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_1\right)} e^{\frac{-\eta}{2\delta_1}\tan^{-1}(s)}$$
$$R_n^{\left(-\delta_1, \frac{-\eta}{\delta_1}\right)} \left(\frac{\sqrt{1-\lambda r^2}}{\sqrt{\lambda} r}\right) \cos^{2\rho}\left(\frac{\theta}{2}\right) \left(1 - \cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma} \times$$
$$F(-l, l+1+(m^2+\alpha-\beta+\gamma)^{1/2} + (m^2+\alpha+\beta+\gamma)^{1/2}; 1+(m^2+\alpha-\beta+\gamma)^{1/2}; \cos^2\left(\frac{\theta}{2}\right)) \tag{4.49}$$

Where $\delta_2 = \sqrt{\frac{1}{2} - E_\theta + \frac{2\mu^2 D_r}{\hbar^2} + k_2}, \eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$
$\rho = \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}$,$\sigma = \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}$

For the potential $\mathbf{V}_2(r,\theta) = \boldsymbol{\mu}\left[-\frac{H}{r} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta + \beta\cos\theta + \gamma)\sin^{-2}\theta\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_1(r,\theta)$ above when we put $D_r \longrightarrow 0$ so

deSitter Space The final expression of energy is

$$E_{n_r,l,m} = -\frac{\mu^3 H^2}{2\hbar^2}\left[n_r + \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2}\right]^{-2} - \frac{\lambda\hbar^2}{2m}\left(n_r^2 + \alpha - \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2 - 1\right) \quad (4.50)$$

$n_r = 0, 1, 2, ..., l = 0, 1, 2, ...$and $m = 0, \pm 1, \pm 2, ...$

The wave function of our system is

$$\psi_1 = N\exp(im\varphi)\left(1 - \frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)}\left(1 + \frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)} P_{n_r}^{\left(-\delta_1-\frac{\eta}{2\delta_1},-\delta+\frac{\eta}{2\delta_1}\right)}\left(\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma} \times F(-l, l+1+(m^2+\alpha-\beta+\gamma)^{1/2} + (m^2+\alpha+\beta+\gamma)^{1/2}; 1 + (m^2+\alpha-\beta+\gamma)^{1/2}; \cos^2\left(\frac{\theta}{2}\right)) \quad (4.51)$$

Where$\delta_1 = \sqrt{\frac{1}{2} - \left(E_\theta^{(2m)} - \frac{2\mu^2 D_r}{\hbar^2}\right) - k_1}, \eta = \frac{2\mu^2 H}{\hbar^2\sqrt{\lambda}}$
$\rho = \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}$,$\sigma = \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}$

Anti deSitter Space The final expression of energy is

$$E_{n_r,l,m} = -\frac{\mu^3 H^2}{2\hbar^2}\left[n_r + \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2}\right]^{-2} + \frac{\lambda\hbar^2}{2m}\left(n_r^2 + \alpha - \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2 - 1\right) \quad (4.52)$$

$n_r = 0, 1, 2, ..., l = 0, 1, 2, ...$and $m = 0, \pm 1, \pm 2, ...$

The wave function of our system is

$$\psi_1 = N \exp(im\varphi) \left(1 + \frac{1 - \lambda r^2}{\lambda r}\right)^{\frac{1}{2}\left(\frac{1}{2} - \delta_1\right)} e^{\frac{-\eta}{2\delta_1} \tan^{-1}(s)}$$
$$R_n^{\left(-\delta_1, \frac{-\eta}{\delta_1}\right)} \left(\frac{\sqrt{1 - \lambda r^2}}{\sqrt{\lambda} r}\right) \cos^{2\rho}\left(\frac{\theta}{2}\right) \left(1 - \cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma} \times$$
$$F(-l, l + 1 + (m^2 + \alpha - \beta + \gamma)^{1/2} + (m^2 + \alpha + \beta + \gamma)^{1/2}; 1 + (m^2 + \alpha - \beta + \gamma)^{1/2}; \cos^2\left(\frac{\theta}{2}\right)) \tag{4.53}$$

Where$\delta_2 = \sqrt{\frac{1}{2} - E_\theta + k_2}, \eta = \frac{2\mu^2 H}{\hbar^2 \sqrt{\lambda}}$
$\rho = \frac{1}{2}(m^2 + \alpha - \beta + \gamma)^{1/2}$,$\sigma = \frac{1}{2}(m^2 + \alpha + \beta + \gamma)^{1/2}$

For the deformed **Kratzer +ring shaped** potential

$V_{K+RS}(r, \theta) = \mu \left[-\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\frac{\gamma}{\sin^2\theta}\right)\right]$,the deformed energy in Hartree units system is

In deSitter space

$$E_{K+RS}(n, l, m) =$$
$$-\frac{1}{2}\left(n - l - m - \frac{1}{2} + \sqrt{\frac{1}{4} + 2D_r + \left[l + (m^2 + \gamma)^{1/2} + \frac{1}{2}\right]^2}\right)^{-2} -$$
$$\frac{\lambda}{2}\left((n - 1 - l - m)^2 - \left[l + (m^2 + \gamma)^{1/2} + \frac{1}{2}\right]^2 - 2D_r - 1\right) \tag{4.54}$$

In anti deSitter space

$$E_{K+RS}(n, l, m) =$$
$$-\frac{1}{2}\left(n - l - m - \frac{1}{2} + \sqrt{\frac{1}{4} + 2D_r + \left[l + (m^2 + \gamma)^{1/2} + \frac{1}{2}\right]^2}\right)^{-2} +$$
$$\frac{\lambda}{2}\left((n - 1 - l - m)^2 - \left[l + (m^2 + \gamma)^{1/2} + \frac{1}{2}\right]^2 - 2D_r - 1\right) \tag{4.55}$$

in order to show the effects of the deformed Heisenberg algebra leading to EUP on the bound states of this potential in 3 dimensional space we plotted the variation of the deformed energy in terms of the parameter of deformation λ $(Figures\ 4.1, 4.2)$

The critical values of the deformation parameter which cancels the bound state is

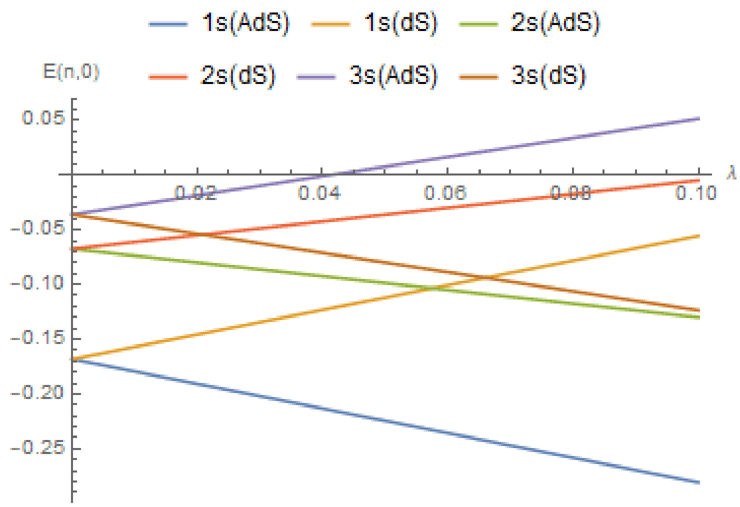

Figure 4.1: $E_{n,0,0}(\lambda)$ of 3D Kratzer potential for $n = 1, 2$ and 3 in dS and AdS cases

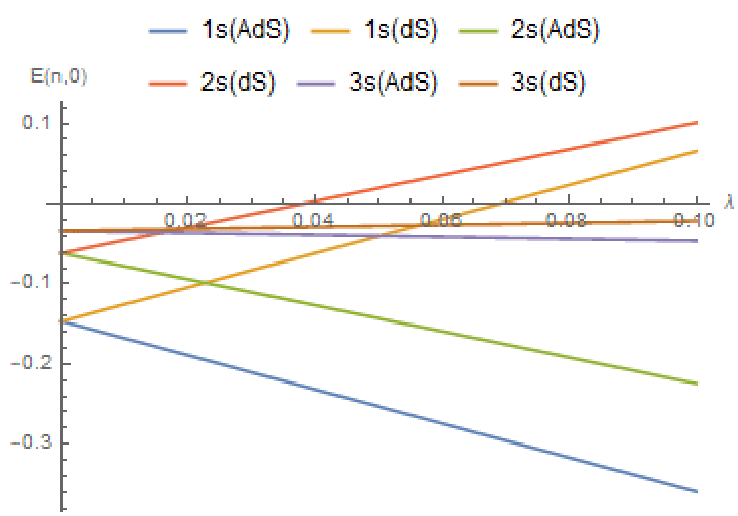

Figure 4.2: $E_{n,0,0}(\lambda)$ of 3D Kratzer + ring-shaped potential for $n = 1, 2$ and 3 in dS and AdS cases

$\lambda_C(m=0)$	I=0	I=1	I=2	l=3	l=4
n=1	0.7116	╲	╲	╲	╲
n=2	0.0704	0.0224	╲	╲	╲
n=3	0.0123	0.0106	0.0055	╲	╲
n=4	0.0037	0.0044	0.0033	0.0020	╲
n=5	0.0015	0.0019	0.0018	0.0013	0.0009

Table 4.1: Critical values for the levels n = 2, 3, 4 and 5 in AdS case for the 3D Kratzer+ring-shaped potential

$\lambda_C(m=0)$	l=0	l=1	l=2	l=3	l=4
n=1	╲	╲	╲	╲	╲
n=2	0.4709	0.0596	╲	╲	╲
n=3	0.0200	0.0204	0.0060	╲	╲
n=4	0.0050	0.0059	0.0036	0.0015	╲
n=5	0.0018	0.0021	0.0018	0.0011	0.0005

Table 4.2: Critical values for the levels n = 2, 3, 4 and 5 in AdS case for the 3D Kratzer potential

$$\lambda_c\left(n,l,m\right)=\frac{\left(n-l-m-\frac{1}{2}+\sqrt{\frac{1}{4}+2D_r+\left[l+(m^2+\gamma)^{1/2}+\frac{1}{2}\right]^2}\right)^{-2}}{\left(\left(n-l-m-1\right)^2-\left[l+(m^2+\gamma)^{1/2}+\frac{1}{2}\right]^2-2D_r-1\right)} \tag{4.56}$$

In $(Table\ 4.1)$ some critical values $\lambda_c\left(n,l,0\right)$ for the Kratzer+ring-shaped potential and $(Table\ 4.2)$ for Kratzer+ring-shaped potential in Hartree system of units and for $\left(D_r=0.5,\gamma=1\right)$,

that to compare the influence of ring-shaped potential to the effect of the deformation parameter

The value of $\lambda_f\left(n,l,m\right)$ that causes this inversion between the upper levels and the fundamental one is

$$\lambda_f\left(n,l,m\right)=\frac{2\left(n-l-m-\frac{1}{2}+\sqrt{\frac{1}{4}+2D_r+\left[l+(m^2+\gamma)^{1/2}+\frac{1}{2}\right]^2}\right)^2-1}{\left(n-l-m-\frac{1}{2}+\sqrt{\frac{1}{4}+2D_r+\left[l+(m^2+\gamma)^{1/2}+\frac{1}{2}\right]^2}\right)^2}$$
$$\times\frac{1}{\left(\left(n-1-l-m\right)^2-\left[l+(m^2+\gamma)^{1/2}+\frac{1}{2}\right]^2-2D_r-1\right)} \tag{4.57}$$

In $(Tables\ 4.4, 4.3)$, we give some numerical values of $\lambda_f\left(n,l,0\right)$

$\lambda_f(m=0)$	l=0	l=1	l=2	l=3	l=4
n=1	7.2883	$\diagdown$	$\diagdown$	$\diagdown$	$\diagdown$
n=2	1.5295	0.4481	$\diagdown$	$\diagdown$	$\diagdown$
n=3	0.4582	0.3703	0.1895	$\diagdown$	$\diagdown$
n=4	0.2124	0.2380	0.1744	0.1075	$\diagdown$
n=5	0.1215	0.1490	0.1385	0.1025	0.0698

Table 4.3: Critical values for the levels n = 2, 3, 4 and 5 in dS case for the 3D Kratzer+ring-shaped potential

$\lambda_f(m=0)$	l=0	l=1	l=2	l=3	l=4
n=1	6.6553	$\diagdown$	$\diagdown$	$\diagdown$	$\diagdown$
n=2	1.4922	0.4287	$\diagdown$	$\diagdown$	$\diagdown$
n=3	0.4536	0.3641	0.1858	$\diagdown$	$\diagdown$
n=4	0.2113	0.2360	0.1728	0.1064	$\diagdown$
n=5	0.1211	0.1483	0.1377	0.1019	0.0694

Table 4.4: Critical values for the levels n = 2, 3, 4 and 5 in dS case for the 3D Kratzer potential

Case2 $V_3(r,\theta)=\mu\left[kr^2+\frac{D_r}{r^2}+\frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta+\beta\cos\theta+\gamma)\sin^{-2}\theta\right]$

Solution of Angular Equation The constant of separation and the angular part of wave function is the same of case1

Solution of Radial Equation So in this case the radial equation 4.9 is

$$\left[\left(1+\tau\lambda r^2\right)\frac{d^2}{dr^2}+\tau\lambda r\frac{d}{dr}+\frac{2\left(1+\tau\lambda r^2\right)}{r}\frac{d}{dr}-\right.$$
$$\left.\frac{\left(E_\theta-\frac{2\mu^2}{\hbar^2}D_r+\frac{1}{4}\right)\left(1+\tau\lambda r^2\right)}{r^2}-\frac{2\mu^2}{\hbar^2}K\frac{r^2}{1+\tau\lambda r^2}+\left(\frac{2\mu E}{\hbar^2}+\frac{\tau\lambda}{2}\right)\right]R(r)=0 \tag{4.58}$$

In order to solve this radial equation we use the following transformations

$$y=\sqrt{1+\tau\lambda r^2}\Longrightarrow r^2=\frac{y^2-1}{\tau\lambda} \tag{4.59}$$

We have to calculate the derivatives with respect to a new variable y,

the first derivative is

$$\frac{d}{dr}=\frac{dy}{dr}\frac{d}{dy}=\frac{\tau\lambda r}{\sqrt{1+\tau\lambda r^2}}\frac{d}{dy}=\frac{\sqrt{\tau\lambda}\sqrt{y^2-1}}{y}\frac{d}{dy} \tag{4.60}$$

The second derivative is

$$\frac{d^2}{dr^2}=\frac{d}{dr}\left(\frac{\tau\lambda r}{\sqrt{1+\tau\lambda r^2}}\frac{d}{dy}\right)=\frac{\tau\lambda}{\left(1+\tau\lambda r^2\right)^{\frac{3}{2}}}\frac{d}{dy}+\left(\frac{\tau\lambda r}{\sqrt{1+\tau\lambda r^2}}\right)^2\frac{d^2}{dy^2} \tag{4.61}$$

When we substitute the expression of r by y we find

$$\frac{d^2}{dr^2}=\frac{d}{dr}\left(\frac{\tau\lambda r}{\sqrt{1+\tau\lambda r^2}}\frac{d}{dy}\right)=\frac{\tau\lambda}{y^3}\frac{d}{dy}+\frac{\tau\lambda\left(y^2-1\right)}{y^2}\frac{d^2}{dy^2} \tag{4.62}$$

by using the derivatives of 3.102 and 3.103 the equation 4.58 becomes

$$\left[\left(y^2-1\right)\frac{d^2}{dy^2}+3y\frac{d}{dy}-\frac{\left(E_\theta-\frac{2\mu^2}{\hbar^2}D_r+\frac{1}{4}\right)y^2}{y^2-1}-\frac{2\mu^2}{\hbar^2}K\frac{y^2-1}{\tau^2\lambda^2y^2}+\left(\frac{2\mu E}{\tau\lambda\hbar^2}+\frac{1}{2}\right)\right]R(r)=-0 \tag{4.63}$$

In order to writ the last equation 4.63 as a Nikiforov–Uvarov equation we have to use the following transformation

$$R\left(y\right)=y^{\upsilon}g\left(y\right) \tag{4.64}$$

Thus the equation 4.63 becomes

$$\left(y^2-1\right)\left[\upsilon\left(\upsilon-1\right)y^{\upsilon-2}g\left(y\right)+2\upsilon y^{\upsilon-1}\frac{d}{dy}g\left(y\right)+y^{\upsilon}\frac{d^2}{dy^2}g\left(y\right)\right]+3y\left[\upsilon y^{\upsilon-1}g\left(y\right)+y^{\upsilon}\frac{d}{dy}g\left(y\right)\right]+ \tag{4.65}$$
$$\frac{y^2}{\left(y^2-1\right)}\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}+\frac{1}{4}\right)y^{\upsilon}g\left(y\right)-\frac{2\mu^2K}{\hbar^2}\frac{\left(y^2-1\right)}{\tau^2\lambda^2y^2}y^{\upsilon}g\left(y\right)+\left(\frac{2\mu E}{\tau\lambda\hbar^2}+\frac{1}{2}\right)y^{\upsilon}g\left(y\right)=0$$

We divide by y^{υ},so

$$\left(y^2-1\right)\frac{d^2}{dy^2}g\left(y\right)+\left(\left(2\upsilon+3\right)y-\frac{2\upsilon}{y}\right)\frac{d}{dy}g\left(y\right)+\frac{\left(y^2-1\right)}{y^2}\upsilon\left(\upsilon-1\right)g\left(y\right)+3\upsilon g\left(y\right)+ \tag{4.66}$$
$$\frac{y^2}{\left(y^2-1\right)}\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}+\frac{1}{4}\right)g\left(y\right)-\frac{2\mu^2K}{\hbar^2}\frac{\left(y^2-1\right)}{\tau^2\lambda^2y^2}g\left(y\right)+\left(\frac{2\mu E}{\tau\lambda\hbar^2}+\frac{1}{2}\right)g\left(y\right)=0$$

We put $\upsilon\left(\upsilon-1\right)-\frac{2\mu^2K}{\hbar^2\tau^2\lambda^2}=0$ this require that $\upsilon=\upsilon_1=\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\tau^2\lambda^2}}$ or $\upsilon=\upsilon_2=\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\tau^2\lambda^2}}$ and the equation 4.66 becomes

$$\left[\left(1-y^2\right)\frac{d^2}{dy^2}+\left(\frac{2\upsilon}{y}-\left(2\upsilon+3\right)y\right)\frac{d}{dy}+\right.$$
$$\left.\frac{y^2}{\left(1-y^2\right)}\left(E_\theta-\frac{2\mu^2D_r}{\hbar^2}+\frac{1}{4}\right)-\left(\frac{2\mu E}{\tau\lambda\hbar^2}+\frac{1}{2}\right)-3\upsilon\right]g\left(y\right)=0 \tag{4.67}$$

The accepted value of υ is the second solution because, from the expression of $R(r)$, the function $g(y)$should be nonsingular at ;$y = \pm 1$

we note that the equation 3.109 possesses three singular points $y = 0, \pm 1$ and to reduce it to a class of known differential equation with a polynomial solution, we use a new variable

$$s = 2y^2 - 1 \Longrightarrow y = \sqrt{\frac{s+1}{2}} \tag{4.68}$$

Now we have to calculate the derivatives with respect to a new variable s

the first derivative with respect to y in terms of s is

$$\frac{d}{dy} = 2\sqrt{2(s+1)}\frac{d}{ds} \tag{4.69}$$

the second derivative with respect to y in terms of s is

$$\frac{d^2}{dy^2} = 4\frac{d}{ds} + 8(s+1)\frac{d^2}{ds^2} \tag{4.70}$$

We use the last derivatives in equation 3.109 we get

$$\begin{gathered}\left[\left(1-s^2\right)\frac{d^2}{ds^2} + ((\upsilon-1)-(\upsilon+2)s)\frac{d}{ds} + \right.\\ \left.\frac{1+s}{4(1-s)}\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2} + \frac{1}{4}\right) - \frac{1}{4}\left(\frac{2\mu E}{\tau\lambda\hbar^2} + \frac{1}{2}\right) - \frac{3}{4}\upsilon\right] g(s) = 0\end{gathered} \tag{4.71}$$

We divide by $(1-s^2)$ this yield the following equation

$$\begin{gathered}\left[\frac{d^2}{ds^2} + \frac{((\upsilon-1)-(\upsilon+2)s)}{(1-s^2)}\frac{d}{ds} + \frac{1+s}{4(1-s)(1-s^2)}\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2} + \frac{1}{4}\right)\right.\\ \left.-\frac{1}{4(1-s^2)}\left(\frac{2\mu E}{\tau\lambda\hbar^2} + \frac{1}{2}\right) - \frac{3\upsilon}{4(1-s^2)}\right] g(s) = 0\end{gathered} \tag{4.72}$$

We put $\frac{2\mu E}{4\tau\lambda\hbar^2} + \frac{1}{2} - 3\upsilon = \varepsilon$ that give us

$$\left[\frac{d^2}{ds^2} + \frac{((\upsilon-1)-(\upsilon+2)s)}{(1-s^2)}\frac{d}{ds} - \frac{(\varepsilon)(1-s^2)}{4(1-s^2)^2} + \frac{(1+s)^2}{4(1-s^2)^2}\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2} + \frac{1}{4}\right)\right] g(s) = 0 \tag{4.73}$$

After some simplification the last equation.4.73 becomes

$$\begin{gathered}\left[\frac{d^2}{ds^2} + \frac{(\upsilon-1)-(\upsilon+2)s}{(1-s^2)}\frac{d}{ds} + \right.\\ \left.\frac{\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2} + \frac{1}{4} + \varepsilon\right)s^2 + 2\left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2} + \frac{1}{4}\right)s + \left(E_\theta - \frac{2\mu^2 D_r}{\hbar^2} + \frac{1}{4} - \varepsilon\right)}{4(1-s^2)^2}\right] g(s) = 0\end{gathered} \tag{4.74}$$

To determine polynomials we compare equation 4.74 with equation 3.18 ,so

$$\begin{gathered}\sigma(s)=\left(1-s^2\right), \quad \tilde{\tau}(s)=(\upsilon-1)-(\upsilon+2)s \text{ and}\\ \tilde{\sigma}(s)=\frac{1}{4}\left[\left(E_\theta-\frac{2\mu^2 D_r}{\hbar^2}+\frac{1}{4}+\varepsilon+3\upsilon\right)s^2+\right.\\ \left.2\left(E_\theta-\frac{2\mu^2 D_r}{\hbar^2}+\frac{1}{4}\right)s+\left(E_\theta-\frac{2\mu^2 D_r}{\hbar^2}+\frac{1}{4}-\varepsilon-3\upsilon\right)\right]\end{gathered} \tag{4.75}$$

Substituting them into Equation.3.24 $\pi(s)=\left(\frac{\sigma^{`}-\tilde{\tau}}{2}\right)\pm\sqrt{\left(\frac{\sigma^{`}-\tilde{\tau}}{2}\right)^2-\tilde{\sigma}+\sigma k}$ we obtain

$$\begin{gathered}\pi(s)=\frac{\upsilon s-(\upsilon-1)}{2}\pm\\ \frac{1}{2}\sqrt{\begin{gathered}\left(\upsilon^2-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}-\frac{1}{4}-\varepsilon-4k\right)s^2+2\left(-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}-\frac{1}{4}-\upsilon(\upsilon-1)\right)s\\ +\left((\upsilon-1)^2-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}-\frac{1}{4}+\varepsilon+4k\right)\end{gathered}}\end{gathered} \tag{4.76}$$

The value of k is obtained from the condition that quadratic expression under the square root in 3.118 has to be completely square of first degree of polynomial therefore the discriminate of the quadratic expression under the square root that has to be zero and $\pi(s)$ can be written as

$$\begin{gathered}\pi(s)=\frac{\upsilon s-(\upsilon-1)}{2}\pm\frac{1}{2}\sqrt{\left(\upsilon^2-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}-\frac{1}{4}-\varepsilon-4k\right)}\\ \left(s-\frac{\left(E_\theta-\frac{2\mu^2 D_r}{\hbar^2}+\frac{1}{4}+\upsilon(\upsilon-1)\right)}{\left(\upsilon^2-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}-\frac{1}{4}-\varepsilon-4k\right)}\right)\end{gathered} \tag{4.77}$$

Therefore the discriminate of the quadratic expression under the square root that has to be zero is given as

$$\begin{gathered}\left(-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}-\frac{1}{4}-\upsilon(\upsilon-1)\right)^2-\\ \left(\upsilon^2-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}-\frac{1}{4}-\varepsilon-4k\right)\left((\upsilon-1)^2-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}-\frac{1}{4}+\varepsilon+4k\right)=0\end{gathered} \tag{4.78}$$

We writ the last equation as algebraic equation of second degree with respect to k

$$16k^2+4(2\varepsilon-2\upsilon+1)k-4\left(-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}-\frac{1}{4}\right)(2\upsilon-1)^2+\varepsilon^2-(2\upsilon-1)\varepsilon=0 \tag{4.79}$$

Now to find k we have to solve this equation ,the discriminate of this equation is $\triangle$

$$\triangle = 64\left(2\upsilon-1\right)^{2}\left(-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}\right) \tag{4.80}$$

So we have to values for k

$$k_{1}=\frac{1}{8}\left[\left(2\upsilon-1-2\varepsilon\right)-2\left(2\upsilon-1\right)\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right] \tag{4.81}$$

And

$$k_{2}=\frac{1}{8}\left[\left(2\upsilon-1-2\varepsilon\right)+2\left(2\upsilon-1\right)\sqrt{\left(-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}\right)}\right] \tag{4.82}$$

we substitute by k in equation 4.79 to find $\pi\left(s\right)$

For $k_1$we find two values of $\pi\left(s\right)$ as bellow

$$\pi_{1}\left(s\right)=\left(\upsilon-\frac{1}{4}+\frac{1}{2}\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)s-\left(\upsilon-\frac{3}{4}\right)+\frac{1}{2}\left(\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right) \tag{4.83}$$

And

$$\pi_{2}\left(s\right)=\left(\frac{1}{4}-\frac{1}{2}\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right)s+\frac{1}{4}-\frac{1}{2}\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}} \tag{4.84}$$

We choose $\pi_{2}\left(s\right)$ for the limit of ordinary space and use it to calculate $\tau^{`}$

$$\begin{aligned}\tau\left(s\right) &= \left(\upsilon-1\right)-\left(\upsilon+2\right)s+2\pi\left(s\right)\Longrightarrow\tau^{`}=-\left(\upsilon+2\right)+2\pi^{`}\\ &\Longrightarrow \tau^{`}=-\upsilon-\frac{3}{2}-\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\end{aligned} \tag{4.85}$$

from the relation $\Lambda_{n}+n_{r}\tau^{`}+\frac{n(n-1)\sigma^{``}}{2}=0$ we have

$$\Lambda=\Lambda_{n}=-n_{r}\left[-\upsilon-\frac{3}{2}-\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right]+n_{r}\left(n_{r}-1\right) \tag{4.86}$$

When we use the expression $\pi\left(s\right)=\pi_{2}\left(s\right)$ of in the equation $k=\Lambda-\pi^{`}\left(s\right)$

$$\Lambda=k_{1}+\left(\frac{1}{4}-\frac{1}{2}\sqrt{-E_{\theta}+\frac{2\mu^{2}D_{r}}{\hbar^{2}}}\right) \tag{4.87}$$

We get the energy eigenvalues from equations 4.86 and 4.87 when we use $\varepsilon=\frac{2\mu E}{4\tau\lambda\hbar^{2}}+\frac{1}{2}-3\upsilon$

and $\upsilon = \upsilon_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8\mu^2 K}{\hbar^2\tau^2\lambda^2}}$

$$E = \frac{\hbar}{\mu}\sqrt{\hbar^2\tau^2\lambda^2 + 2\mu^2 K}\left(2\left(1-n_r\right) - \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)$$
$$+\frac{2\tau\lambda\hbar^2}{\mu}\left(-4n_r^2 - 4n_r + 2 - \left(4n_r+2\right)\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right) \tag{4.88}$$

Now we have to writ the expression of the radial wave functions as $R\left(s\right) = g(s)$, we first get $g(s) = \phi\left(s\right)\rho\left(s\right)$

We substitute by the expression of π_1 and $\sigma\left(s\right) = \left(1 - s^2\right)$ in equation 3.57 to find $\phi\left(s\right)$ as

$$\pi\left(s\right) = \pi_2\left(s\right) = \sigma\left(s\right)\frac{d}{ds}\left(\ln\phi\left(s\right)\right) \Longrightarrow \phi\left(s\right) = Exp\left(\int\frac{\pi\left(s\right)}{\sigma\left(s\right)}ds\right)$$
$$\Longrightarrow \phi\left(s\right) = Exp\left(\int\frac{\pi\left(s\right)}{\sigma\left(s\right)}ds\right) \tag{4.89}$$

We substitute by the expression of $\pi\left(s\right) = \left(\frac{1}{4} - \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)s + \frac{1}{4} - \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}$ and $\sigma\left(s\right)$ we find

$$\phi\left(s\right) = Exp\left(\int\frac{\left(\frac{1}{4} - \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)s + \left(\frac{1}{4} - \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)}{\left(1-s^2\right)}ds\right) \tag{4.90}$$

After the calculation of the integral we obtain

$$\phi\left(s\right) = Exp\left(\left(\frac{1}{4} - \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)\left(-\frac{1}{2}\ln\left(1-s^2\right)\right)\right.$$
$$\left.+\left(\frac{1}{4} - \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)\left(-\frac{1}{2}\ln\left(1-s\right) + \frac{1}{2}\ln\left(1+s\right)\right)\right) \tag{4.91}$$

So the function $\phi\left(s\right)$ is

$$\phi\left(s\right) = \left(1+s\right)^{\frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)} \tag{4.92}$$

We use 3.26 to find the weight function $\rho\left(s\right)$

$$\frac{d}{ds}\left[\sigma(s)\rho(s)\right]=\tau(s)\rho(s)\Longrightarrow\sigma(s)\frac{d\rho(s)}{ds}+\frac{d\sigma(s)}{ds}\rho(s)=\left[\tau(s)\rho(s)\right]\Longrightarrow$$
$$\sigma(s)\frac{d\rho(s)}{\rho(s)ds}+\frac{d\sigma(s)}{ds}=\left[\tau(s)\right]\Longrightarrow\int\frac{d\rho(s)}{\rho(s)}=\int\left(\frac{\tau(s)}{\sigma(s)}-\frac{d\sigma(s)}{\sigma(s)ds}\right)ds \tag{4.93}$$

When we compute the integral we get

We use the expression of $\tau(s)$ from equation 3.126 and $\sigma(s)$ to find the weight function $\rho(s)$ from equation 3.62

$$\rho(s)=\exp$$
$$\left[\int\left(\frac{(\upsilon-1)-(\upsilon+2)s+\left(\frac{1}{2}-1\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)s+\frac{1}{2}-1\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}+2s}{(1-s^2)}\right)ds\right] \tag{4.94}$$

After the calculation of the integral we find

$$\rho(s)=\exp\left[\frac{1}{2}\left(\frac{1}{2}-1\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}-\upsilon\right)\left(-\ln\left(1-s^2\right)\right)+\right.$$
$$\left.\frac{1}{2}\left(\upsilon-\frac{1}{2}-1\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)\left(-\ln(1-s)+\ln(1+s)\right)\right] \tag{4.95}$$

So

$$\rho(s)=(1-s)^{\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}}(1+s)^{\upsilon-\frac{1}{2}} \tag{4.96}$$

the $y_n(s)$part is given by Rodrigues relation

$$y_n(s)=\frac{C_n}{\rho(s)}\frac{d^n}{ds^n}\left[\left(1-s^2\right)^n\rho(s)\right] \tag{4.97}$$

We substitute by the expression of $\rho(s)$ we find

$$y_n(s)=\frac{C_n}{\rho(s)}\frac{d^n}{ds^n}\left[\left(1-s^2\right)^n(1-s)^{\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}}(1+s)^{\upsilon-\frac{1}{2}}\right]$$

After the calculation of the integral we find

$$\rho(s)=\frac{(1-s)^{\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}}}{(1+s)^{\left(\upsilon-\frac{1}{2}\right)}} \tag{4.98}$$

the $y_n(s)$part is given by Rodrigues relation

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} \left[(\sigma(s))^n \rho(s) \right] \tag{4.99}$$

Where$\rho(s) = \frac{(1-s)^{\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}}}{(1+s)^{\left(\upsilon - \frac{1}{2}\right)}}$. and $\sigma(s) = (1 - s^2)$ equation 4.99 stands for the Romanovski polynomials as

$$y_n(s) \equiv p_n^{\left(\frac{1}{2}-\upsilon, \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)}(s) =$$
$$\frac{C_n (1+s)^{\left(\upsilon - \frac{1}{2}\right)}}{(1-s)^{\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}}} \frac{d^n}{ds^n} \left[(1-s)^{n + \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}} (1+s)^{n - \left(\upsilon - \frac{1}{2}\right)} \right] \tag{4.100}$$

From equation 4.92 and 4.100 the function $g(s)$ is

$$g(s) = \phi(s)\, y_n(s) = (1+s)^{\frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)} p_n^{\left(\frac{1}{2}-\upsilon, \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)}$$

Hence, $R(s)$ can be written in the following form $R(y) = y^{\upsilon} g(y)\,, s = 2y^2 - 1$

$$R(y) = C_n \left(2y^2\right)^{\frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)} p_n^{\left(\frac{1}{2}-\upsilon, \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)} y^{\upsilon} \tag{4.101}$$

We have $\upsilon = \upsilon_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8\mu^2 K}{\hbar^2 \tau^2 \lambda^2}}$ and $y = \sqrt{1 + \tau\lambda r^2}$ so the radial wave function can be written as

$$R(r) = C_n \left(2 + 2\tau\lambda r^2\right)^{\left(\frac{1}{8} - \frac{1}{4}\sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)}$$
$$p_n^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\tau^2\lambda^2}}, \sqrt{-E_\theta + \frac{2\mu^2 D_r}{\hbar^2}}\right)} \left(\sqrt{1 + \tau\lambda r^2}\right)^{\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\tau^2\lambda^2}}} \tag{4.102}$$

C_n is the normalization constant

Energy and Wave Function

deSitter Space We substitute the constant of separation 2.36 in the expression of energy 4.88 ,we find the final expression of energy as

$$E = \frac{\hbar}{2\mu}\sqrt{\hbar^2\lambda^2 + 8\mu^2 K}$$
$$\left(2n_r + 1 + \sqrt{-\alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2 + \frac{2\mu^2 D_r}{\hbar^2} + \frac{1}{4}}\right.$$
$$-\frac{2\lambda\hbar^2}{\mu}\left(-4n_r^2 - 4n_r + 2 - (4n_r+2)\right.$$
$$\left.\sqrt{-\alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2 + \frac{2\mu^2 D_r}{\hbar^2}}\right) \tag{4.103}$$

The radial wave function is

$$\psi_2 = N\exp\left(im\varphi\right)\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma}\left(2+2\lambda r^2\right)^{\left(\frac{1}{8}-\frac{1}{4}\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}\right)}$$
$$\left(\sqrt{1+\lambda r^2}\right)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\lambda^2}}} p_n^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\lambda^2}},\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}\right)}\times$$
$$F(-l, l+1+(m^2+\alpha-\beta+\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2}; 1+(m^2+\alpha-\beta+\gamma)^{1/2}; \cos^2\left(\frac{\theta}{2}\right)) \tag{4.104}$$

$n_r = 0, 1, 2, ..., \ l = 0, 1, 2, ...$and $m = 0, \pm1, \pm2, ...$

Where$\rho = \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}$,$\sigma = \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}$

Anti- deSitter Space We substitute the constant of separation 2.36 in the expression of energy 4.88 ,we find the final expression of energy as

$$E = \frac{\hbar}{2\mu}\sqrt{\hbar^2\lambda^2 + 8\mu^2 K}$$
$$\left(2n_r + 1 + \sqrt{-\alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2 + \frac{2\mu^2 D_r}{\hbar^2} + \frac{1}{4}}\right.$$
$$+\frac{2\lambda\hbar^2}{\mu}\left(-4n_r^2 - 4n_r + 2 - (4n_r+2)\right.$$
$$\left.\sqrt{-\alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2 + \frac{2\mu^2 D_r}{\hbar^2}}\right) \tag{4.105}$$

The radial wave function is

$$\psi_2 = N\exp(im\varphi)\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma}\left(2-2\lambda r^2\right)^{\left(\frac{1}{8}-\frac{1}{4}\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}\right)}$$
$$\left(\sqrt{1-\lambda r^2}\right)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\lambda^2}}} p_n^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\lambda^2}},\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}\right)}\times$$
$$F(-l,l+1+(m^2+\alpha-\beta+\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2};1+(m^2+\alpha-\beta+\gamma)^{1/2};\cos^2\left(\frac{\theta}{2}\right)) \tag{4.106}$$

$n_r = 0,1,2,...,\ l=0,1,2,...$and $m=0,\pm1,\pm2,...$

Where$\rho=\frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}$,$\sigma=\frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}$

For the potential $V_4(r,\theta)=\frac{\mu}{q}\left[kr^2+\frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^2\theta+\beta\cos\theta+\gamma)\sin^{-2}\theta\right]$ we deduce the energy and wave function of this case from the energy and wave function of $V_3(r,\theta)$ when we put $D_r\longrightarrow 0$ so

deSitter Space the final expression of energy as

$$E=\frac{\hbar}{2\mu}\sqrt{\hbar^2\lambda^2+8\mu^2K}$$
$$\left(2n_r+1+\sqrt{-\alpha+\left[l+\frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}+\frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}+\frac{1}{2}\right]^2+\frac{1}{4}}\right)+ \tag{4.107}$$

$$\frac{2\lambda\hbar^2}{\mu}\left(-4n_r^2-4n_r+2-(4n+2)\right.$$
$$\left.\sqrt{-\alpha+\left[l+\frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}+\frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}+\frac{1}{2}\right]^2}\right) \tag{4.108}$$

The wave function of our system

$$\psi_2 = N\exp(im\varphi)\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma}\left(2+2\lambda r^2\right)^{\left(\frac{1}{8}-\frac{1}{4}\sqrt{-E_\theta}\right)}$$
$$\left(\sqrt{1+\lambda r^2}\right)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\lambda^2}}} p_n^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2 K}{\hbar^2\lambda^2}},\sqrt{-E_\theta}\right)}\times$$
$$F(-l,l+1+(m^2+\alpha-\beta+\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2};1+(m^2+\alpha-\beta+\gamma)^{1/2};\cos^2\left(\frac{\theta}{2}\right)) \tag{4.109}$$

$n_r = 0,1,2,...,\ l=0,1,2,...$and $m=0,\pm1,\pm2,...$

Where$\rho=\frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}$,$\sigma=\frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}$

Anti- deSitter Space

$$E = \frac{\hbar}{2\mu}\sqrt{\hbar^2\lambda^2 + 8\mu^2 K}$$
$$\left(2n_r + 1 + \sqrt{-\alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2 + \frac{1}{4}}\right) \tag{4.110}$$

$$+\frac{2\lambda\hbar^2}{\mu}\left(-4n_r^2 - 4n_r + 2 - (4n_r+2)\right.$$
$$\left.\sqrt{-\alpha + \left[l + \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2} + \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2} + \frac{1}{2}\right]^2}\right) \tag{4.111}$$

We deduce the wave function of our system $\psi(r,\theta,\varphi) = \exp(im\varphi)\,R(r)\Theta(\theta)$ from the angular part 2.35 and radial part 4.102

$$\psi_2 = N\exp(im\varphi)\cos^{2\rho}\left(\frac{\theta}{2}\right)\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)^{\sigma}\left(2-2\lambda r^2\right)^{\left(\frac{1}{8}-\frac{1}{4}\sqrt{-E_\theta}\right)}$$
$$\left(\sqrt{1-\lambda r^2}\right)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}}} p_n^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}},\sqrt{-E_\theta}\right)}\times$$
$$F(-l, l+1+(m^2+\alpha-\beta+\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2}; 1+(m^2+\alpha-\beta+\gamma)^{1/2}; \cos^2\left(\frac{\theta}{2}\right)) \tag{4.112}$$

$n_r = 0,1,2,...$, $l = 0,1,2,...$and $m = 0,\pm1,\pm2,...$

Where$\rho = \frac{1}{2}(m^2+\alpha-\beta+\gamma)^{1/2}$,$\sigma = \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}$

We summarize the previous results and the results of the rest of the studied potentials in the $(Tables\ 4.5, 4.6)$

E_θ and $\Theta(\theta)$ are shown in$(Tabels\ 1.2, 1.3)$ $(Tables\ 2.1, 2.2)$

4.3 Discussion

We note the same remarks of the two dimensional case in this three dimensional cases. We also notice that the correction deformation affects all energy levels except the ground level (n = 1), which remains not affected by the deformation even for large values of λ.(parameter of deformation)

$V(r,\theta)$	$Space$	E
$\mu\left[-\frac{H}{r}+\frac{f(\theta)}{r^2}\right]$	dS	$-\frac{\mu^3 H^2}{2\hbar^2}\left(n_r+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu^2}{\hbar^2}D_r-E_\theta}\right)^{-2}$ $-\frac{\lambda\hbar^2}{2m}\left(n^2+E_\theta-\frac{2\mu^2}{\hbar^2}D_r-1\right)$
$\mu\left[-\frac{H}{r}+\frac{f(\theta)}{r^2}\right]$	Ads	$-\frac{\mu^3 H^2}{2\hbar^2}\left(n_r+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu^2}{\hbar^2}D_r-E_\theta}\right)^{-2}$ $+\frac{\lambda\hbar^2}{2m}\left(n^2+E_\theta-\frac{2\mu^2}{\hbar^2}D_r-1\right)$
$\mu\left[kr^2+\frac{f(\theta)}{r^2}\right]$	dS	$E=\frac{\hbar}{\mu}\sqrt{\hbar^2\lambda^2+2\mu^2 K}\left(2\left(1-n\right)-\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}\right)$ $+\frac{2\lambda\hbar^2}{\mu}\left(-4n^2-4n+2-\left(4n+2\right)\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}\right)$
$\mu\left[kr^2+\frac{f(\theta)}{r^2}\right]$	AdS	$E=\frac{\hbar}{\mu}\sqrt{\hbar^2\lambda^2+2\mu^2 K}\left(2\left(1-n\right)-\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}\right)$ $-\frac{2\lambda\hbar^2}{\mu}\left(-4n^2-4n+2-\left(4n+2\right)\sqrt{-E_\theta+\frac{2\mu^2 D_r}{\hbar^2}}\right)$

Table 4.5: The expression of deformed energy in 3D space

$V(r,\theta)$	$Space$	ψ
$\mu\left[-\frac{H}{r}+\frac{f(\theta)}{r^2}\right]$	dS	$N\exp(im\varphi)\left(1-\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)^{\frac{1}{4}\left(1-2\delta_1+\frac{\eta}{\delta_1}\right)}\left(1+\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)^{\frac{1}{4}\left(1-2\delta_1-\frac{\eta}{\delta_1}\right)}$ $P_{n_r}^{\left(-\delta_1-\frac{\eta}{2\delta_1},-\delta+\frac{\eta}{2\delta_1}\right)}\left(\frac{\sqrt{1+\lambda r^2}}{\sqrt{\lambda}r}\right)\times\Theta(\theta)$
$\mu\left[-\frac{H}{r}+\frac{f(\theta)}{r^2}\right]$	Ads	$N\exp(im\varphi)\left(1+\frac{1-\lambda r^2}{\lambda r}\right)^{\frac{1}{2}\left(\frac{1}{2}-\delta_1\right)}e^{\frac{-\eta}{2\delta_1}\tan^{-1}(s)}R_n^{\left(-\delta_1,\frac{-\eta}{\delta_1}\right)}\left(\frac{\sqrt{1-\lambda r^2}}{\sqrt{\lambda}r}\right)$ $\times\Theta(\theta)$
$\mu\left[kr^2+\frac{f(\theta)}{r^2}\right]$	dS	$N\exp(im\varphi)(2+2\lambda r^2)^{\left(\frac{1}{8}-\frac{1}{4}\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)}\left(\sqrt{1+\lambda r^2}\right)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}}}$ $p_n^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}},\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)}\Theta(\theta)$
$\mu\left[kr^2+\frac{f(\theta)}{r^2}\right]$	AdS	$N\exp(im\varphi)(2-2\lambda r^2)^{\left(\frac{1}{8}-\frac{1}{4}\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)}\left(\sqrt{1-\lambda r^2}\right)^{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}}}$ $p_n^{\left(-\frac{1}{2}\sqrt{1+\frac{8\mu^2K}{\hbar^2\lambda^2}},\sqrt{-E_\theta+\frac{2\mu^2D_r}{\hbar^2}}\right)}\Theta(\theta)$

Table 4.6: The expression of deformed wave function in 3D space

General Conclusion

The important task of quantum mechanics is to find the exact bound-states solution of the Schrödinger equation in nonrelativistic case and Dirac ,Klein Gordon in relativistic case the aim of our works is reach this task for a kind of potentials which is the non-central potentials in the first part of this thesis we studied some of non central potentials in the ordinary space ,the first chapter devoted to the tow dimensional space Where did we study the potentials analytically in $2D$ space ,in first section of this chapter we have solved the Schrödinger equation for five potential the first four haven't names and the five one is the dipole ,this solvable potentials in $2D$ not much known because Two-dimensional technology is very recent and The usual field of use for this type of potential is chemistry and nuclear physics but this fields are three-dimensional except the dipole have a real applications like grafen ,where we have a good empirical results .So we solved the Schrödinger equation and extracted a well-defined energy and wave function ,but to get bound-states a condition must be fulfilled is $\frac{2\mu^2}{\hbar^2}D_r - E_\theta \geq 0$ where E_θ is a constant of separation of equations contain the parameters of the noncentral potential and D_r is the parameter of kratzer this condition appears in both cases of the Colombian and the oscillator .Since that dipole is our contribution to physics we illustrate it in details ,so for the dipole plus Kratzer potential the spectrum shows that the energies follow mainly the behavior of Mathieu's characteristic parameters and thus the angular moment D_θ, whereas the effect of the radial moment D_r is merely a shift in these energies to larger or smaller values according to its sign. We have showed also that there is an essential condition for bound states to exist, which is: $c_{2m}(4D_\theta) + 8D_r > 0$. This condition imposes a critical value for the angular moment D_θ,depending on the value of m, otherwise the corresponding bound state disappears. These critical values of D_θ depend also on the value of D_r and the negative value of this moment which makes $c_{2m}(4D_\theta) + 8D_r = 0$ is also a critical value for the radial moment. So we see that by increasing,the radial dipole displaces the energies towards the larger values while widening the region of the possible values of the angular moment..The second chapter of first chapter is about the relativistic case ,we took just the spin and pseudospin symmetry ,the eigenfunctions are determined analytically but the energies can only be calculated using graphical methods. Only the spin symmetry has given results corresponding to atomic systems. The behavior of the energies is the same as that of the Schrödinger spectrum but it is shifted because the Schrödinger type equation of the relativistic systems has $2V$ as a potential instead of the potential V in the ordinary

Schrödinger equation. We also note that the critical values of the dipole moments D_r and D_θ depend on the two quantum numbers n and m in the relativistic case instead of just m in the case of non-relativistic systems ,we have found that the angular term removes the degeneracy found in the $exp(im\theta)$ part of the solutions for central potentials. This is equivalent to the effect of a constant magnetic field in 3D systems, where its action removes the degeneracy of the $exp(im\phi)$ solutions too. In both cases,the privileged direction of the interaction (dipole axis in 2D and field direction in 3D) removes the degeneracy that existed due to the isotropy of the action before chapter two is about the three-dimensional non-central potential where we studied four non-central potentials which are general cases of known potentials Hartmann potential Makarov potential ring-shaped potential and double ring potential in both cases non relativistic and relativistic(spin and pseudo spin symmetry),where we found the energy spectrum and the wave functions and the condition to get a bound states is different to the two dimensional potential and it is $\frac{1}{4} + \frac{2\mu^2}{\hbar^2} D_r - E_\theta \geq 0$.

In the second part of this thesis we analytical studied all the potentials of the first part but in deformed space (deSitter and anti deSitter space) by using the position representation of the Extended Uncertainty Principle formulation and the Nikiforov–Uvarov method. For both cases, we obtained the exact eigenenergies and eigenfunctions. The radial wave functions were expressed as associated Jacobi polynomials for de Sitter space and in terms of Romanovski polynomials for anti-de Sitter space.The deformed energy spectrum was written as the ordinary term with an additional correction term The main effect of the deformation parameter λ is an increase in the energies for AdS spaces and a decrease in these energies for dS spaces. for the non-central potential plus Colombian but for the non-central plus oscillator is opposite moreover we deduced a critical values λ_C for the deformed parameter which cancel the energy and critical values λ_f for the deformed parameter which that causes the inversion between the upper levels and the fundamental one

Appendix

.1 Details of Non-Central Potential in 2D Ordinary Space

Case3 $V_5(r,\theta) = \mu\left[-\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\frac{\theta}{2} + \beta\tan\frac{\theta}{2} + \gamma\right)\right]$

For this case the angular equation 1.16 becomes

$$\frac{d^2\Theta}{d\theta^2} - \left(\alpha\tan^2\frac{\theta}{2} + \beta\tan\frac{\theta}{2} + \gamma\right)\Theta - E_\theta\Theta = 0 \tag{113}$$

To solve this equation we make the following substitutions:

$$z = -e^{i\theta} \tag{114}$$

and

$$\Theta = z^\rho(1-z)^\sigma T \tag{115}$$

Now we have to compute all parts of the equation 113 by the new variable z and the new function$T$

From the equation 114 we deduce the following relation

$$z = -e^{i\theta} \Longrightarrow \theta = -i\ln(-z) \tag{116}$$

And its derivative is

$$\frac{dz}{d\theta} = -ie^{i\theta} = iz \tag{117}$$

From the trigonometric relation we find

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}{i\left(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}\right)} = \frac{e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)}{e^{i\frac{\theta}{2}}i\left(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}\right)} = \frac{e^{i\theta}-1}{i\left(e^{i\theta}+1\right)} = \frac{z-1}{i(z+1)} = -i\frac{z-1}{(z+1)} \tag{118}$$

The first derivative of Θ with respect to θ in terms of a new variable z is

$$\frac{d\Theta}{d\theta} = iz\frac{d\Theta}{dz} \tag{119}$$

The second derivative of Θ with respect to θ in terms of a new variable z is

$$\frac{d^2\Theta}{d\theta^2} = -z^2\frac{d^2\Theta}{dz^2} - z\frac{d\Theta}{dz} \tag{120}$$

The first derivative of Θ with respect to the new variable z in terms of a new function is

$$\frac{d\Theta}{dz} = \left(\rho z^{\rho-1}(1-z)^{\sigma} - \sigma z^{\rho}(1-z)^{\sigma-1}\right)T + z^{\rho}(1-z)^{\sigma}\frac{dT}{dz} \tag{121}$$

The second derivative of Θ with respect to the new variable z in terms of a new function is

$$\begin{aligned}\frac{d^2\Theta}{dz^2} &= \left[\left(\rho\left(\rho-1\right)z^{\rho-2}(1-z)^{\sigma} - 2\rho\sigma z^{\rho-1}(1-z)^{\sigma-1} + \sigma\left(\sigma-1\right)z^{\rho}(1-z)^{\sigma-2}\right)+\right]T \\ &+2\left(\rho z^{\rho-1}(1-z)^{\sigma} - \sigma z^{\rho}(1-z)^{\sigma-1}\right)\frac{dT}{dz} + z^{\rho}(1-z)^{\sigma}\frac{d^2T}{dz^2}\end{aligned} \tag{122}$$

By substituting the results 116 to 120 in equation 113 we find a new angular equation for Θ

$$-z^2\frac{d^2\Theta}{dz^2} - z\frac{d\Theta}{dz} - \left(-\alpha\frac{(z-1)^2}{(z+1)^2} - i\beta\frac{z-1}{(z+1)} + \gamma\right)\Theta - E_{\theta}\Theta = 0 \tag{123}$$

By using the equations 121 and 122 ,the last equation 123 becomes

$$\begin{gathered}-z^{\rho+2}(1-z)^{\sigma}\frac{d^2T}{dz^2} - \left[2\left(\rho z^{\rho+1}(1-z)^{\sigma} - \sigma z^{\rho+2}(1-z)^{\sigma-1}\right) + z^{\rho+1}(1-z)^{\sigma}\frac{dT}{dz}\right] - \\ \left[\left(\rho\left(\rho-1\right)z^{\rho}(1-z)^{\sigma} - 2\rho\sigma z^{\rho+1}(1-z)^{\sigma-1} + \sigma\left(\sigma-1\right)z^{\rho+2}(1-z)^{\sigma-2}\right)\right. \\ \left.-\left(-\alpha\tfrac{(z-1)^2}{(z+1)^2} - i\beta\tfrac{z-1}{(z+1)} + \gamma\right)z^{\rho}(1-z)^{\sigma} - E_{\theta}z^{\rho}(1-z)^{\sigma}\right]T = 0\end{gathered} \tag{124}$$

We divide the equation 124 by $z^{\rho+1}(1-z)^{\sigma-1}$we find The following differential equation

$$\begin{gathered}-z(1-z)\frac{d^2T}{dz^2} - \left[2\left(\rho(1-z) - \sigma z\right) + (1-z)\frac{dT}{dz}\right] - \\ \left[\left(\rho\left(\rho-1\right)z^{-1}(1-z) - 2\rho\sigma + \sigma\left(\sigma-1\right)z(1-z)^{-1}\right) - \right. \\ \left.\left(-\alpha\tfrac{(z-1)^2}{(z+1)^2} - i\beta\tfrac{z-1}{(z+1)} + \gamma\right)z^{-1}(1-z) - E_{\theta}z^{-1}(1-z)\right]T = 0\end{gathered} \tag{125}$$

Where we take

$$\rho = \frac{1}{4} + \frac{1}{4}(1 + 4\alpha + 4\beta + 4\gamma)^{1/2} \tag{126}$$

$$\sigma = \frac{1}{2} + \frac{1}{2}(1+16\alpha)^{1/2} \tag{127}$$

The equation 124 becomes a hypergeometric equation type as:

$$z(1-z)\frac{d^2T}{dz^2} + \left[(2\rho+1) - (2\rho+2\sigma+1)z\right]\frac{dT}{dz} - \frac{1}{2}\left[-2\rho\sigma+\sigma+4\alpha-2i\beta\right]T = 0 \tag{128}$$

The last equation is a hypergeometric equation type and its solution is hypergeometric function [3][36]:

$$T = F\left(2\rho, 2\sigma, (2\rho+1); y\right) \tag{129}$$

From the asymptotic behavior of the confluent series ($r \to \infty \Longrightarrow F = 0$) which lead to $T \to 0$ when $r \to \infty$ we find the general condition of quantization :

$$2\rho = -m, m = 0, 1, 2, ... \tag{130}$$

This means that

$$2\rho + m = 0 \tag{131}$$

From 127 we have

$$2\sigma = 2\left(\frac{1}{2} + \frac{1}{2}(1+16\alpha)^{1/2}\right) = 1 + (1+16\alpha)^{1/2}$$

we use 131 we find

$$2\sigma = m + 2\rho + 1 + (1+16\alpha)^{1/2} \tag{132}$$

By using 130 and 132 we can write the hypergeometric function as

$$T = F(-m, m+2\rho+1+(1+16\alpha)^{1/2}; 2\rho+1; z) \tag{133}$$

From the from of the hypergeometric equation [36]

$$\rho^2 = -i\beta + \frac{1}{4}\left\{\left[m + \frac{1}{2} + \frac{1}{2}(1+16\alpha)^{1/2}\right] - 4\beta^2\right\}\left[m + \frac{1}{2} + \frac{1}{2}(1+16\alpha)\right]^{-2} \tag{134}$$

From 126 we have

$$\rho = (-E_\theta + \alpha - i\beta - \gamma)^{1/2} \Longrightarrow \rho^2 = -E_\theta + \alpha - i\beta - \gamma \Longrightarrow E_\theta = \alpha - i\beta - \gamma - \rho^2 \tag{135}$$

This require that the angular energy is

$$E_\theta = \alpha - i\beta - \gamma + i\beta - \frac{1}{4}\left\{\left[m + \frac{1}{2} + \frac{1}{2}(1+16\alpha)^{1/2}\right] - 4\beta^2\right\}\left[m + \frac{1}{2} + \frac{1}{2}(1+16\alpha)\right]^{-2}$$

In another form E_θ can be written as

$$E_\theta = \alpha - \gamma - \frac{\left[m + \frac{1}{2} + \frac{1}{2}(1+16\alpha)^{1/2}\right] - 4\beta^2}{4\left[m + \frac{1}{2} + \frac{1}{2}(1+16\alpha)\right]^2} \tag{136}$$

m is the angular quantification number, $m = 0, 1, 2, ...$

We find the angular wave function when we substitute the function T in the equation $\Theta(z) = z^\rho(1-z)^\sigma T$ as

$$\Theta(z) = z^\rho(1-z)^\sigma F(-m, m + 2\rho + 1 + (1+16\alpha)^{1/2}; 2\rho + 1; z) \tag{137}$$

We have

$$\Theta(\theta) = -e^{i\rho\theta}(1+e^{i\theta})^\sigma F(-m, m + 2\rho + 1 + (1+16\alpha)^{1/2}; 2\rho + 1; -e^{i\theta}) \tag{138}$$

case5:$V_9(r,\theta) = \mu\left[-\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\cot^2\frac{\theta}{2} + \beta\cot\frac{\theta}{2} + \gamma\right)\right]$

For this kind of potential the angular equation 1.16 becomes

$$\frac{d^2\Theta}{d\theta^2} - \left(\alpha\cot^2\frac{\theta}{2} + \beta\cot\frac{\theta}{2} + \gamma\right)\Theta - E_\theta\Theta = 0 \tag{139}$$

To solve this equation we substitute $\theta = \pi - \theta'$,then we have to deduce a new equation for θ'

$$\cot\frac{\theta}{2} = \cot\left(\frac{\pi}{2} - \frac{\theta'}{2}\right) = \tan\frac{\theta'}{2} \Longrightarrow \cot^2\frac{\theta}{2} = \tan^2\frac{\theta'}{2} \tag{140}$$

The first derivative of Θ with respect to θ and The first derivative of Θ with respect to θ' are equal

$$\frac{d\Theta}{d\theta} = \frac{d\theta'}{d\theta}\frac{d\Theta}{d\theta'} = \frac{d\Theta}{d\theta'} \tag{141}$$

The second derivative of Θ with respect to θ and θ' is same also

$$\frac{d^2\Theta}{d\theta^2} = \frac{d^2\Theta}{d\theta'^2} \tag{142}$$

We substitute the equation 140 to 142 the equation 139 for the new variable θ' becomes

$$\frac{d^2\Theta}{d\theta'^2} - \left(\alpha\tan^2\frac{\theta'}{2} + \beta\tan\frac{\theta'}{2} + \gamma\right)\Theta - E_{\theta'}\Theta = 0 \tag{143}$$

Which is the same angular equation of $case2$,then we can deduce the angular wave function and the angular energy just by change θ by θ' in the expression of wave function 138 and energy 136 of $case2$

$$\Theta\left(\theta'\right) = -e^{i\rho\theta'}(1+e^{i\theta'})^{\sigma}F(-m, m+2\rho+1+(1+16\alpha)^{1/2}; 2\rho+1; -e^{i\theta'}) \tag{144}$$

We substitute by $\theta' = \theta + \pi$:

$$\Theta\left(\theta\right) = -e^{i\rho(\theta+\pi)}(1+e^{i(\theta+\pi)})^{\sigma}F(-m, m+2\rho+1+(1+16\alpha)^{1/2}; 2\rho+1; -e^{i(\theta+\pi)}) \tag{145}$$

So the angular function for this case is

$$\Theta\left(\theta\right) = -e^{i\rho\pi}e^{i\rho\theta}(1+e^{i\pi}e^{i\theta})^{\sigma}F(-m, m+2\rho+1+(1+16\alpha)^{1/2}; 2\rho+1; -e^{i\pi}e^{i\theta}) \tag{146}$$

Finally after the simplification the angular wave function of this case is

$$\Theta\left(\theta\right) = (-1)^{i\rho+1}\, e^{i\rho\theta}(1-e^{i\theta})^{\sigma}F(-m, m+2\rho+1+(1+16\alpha)^{1/2}; 2\rho+1; e^{i\theta}) \tag{147}$$

$m = 0, 1, 2, ...$

And the constant of separation E_θ is a same of the energy of $case2$

$$E_\theta = \alpha - \gamma - \frac{\left[m+\frac{1}{2}+\frac{1}{2}\left(1+16\alpha\right)^{1/2}\right] - 4\beta^2}{4\left[m+\frac{1}{2}+\frac{1}{2}\left(1+16\alpha\right)^{1/2}\right]^2} \tag{148}$$

$m = 0, 1, 2, ...$

$\rho = (-E_\theta + \alpha - i\beta - \gamma)^{1/2}$and $\sigma = \frac{1}{2} + \frac{1}{2}(1+16\alpha)^{1/2}$

Case7 $V_{13}\left(r,\theta\right) = \mu\left[-\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)\left(\alpha\tan^2\theta + \beta\tan\theta + \gamma\right)\right]$

For this case the angular equation 1.16 becomes

$$\frac{d^2\Theta}{d\theta^2} - \left(\alpha\tan^2\theta + \beta\tan\theta + \gamma\right)\Theta - E_\theta\Theta = 0 \tag{149}$$

To solve this equation we have to make the following substitutions:

$$z = 1 + e^{i\theta} \tag{150}$$

And

$$\Theta = z^{\rho}(1-z)^{\sigma}T \tag{151}$$

From 150 we deduce the following relations

$$\theta = -i\ln(1-z) \tag{152}$$

$$e^{i2\theta} = (1-z)^2 \tag{153}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{e^{i\theta}-e^{-i\theta}}{i\left(e^{i\theta}+e^{-i\theta}\right)} = \frac{-i\left(e^{i2\theta}-1\right)}{\left(e^{i2\theta}+1\right)} = -\frac{i\left((1-z)^2-1\right)}{\left((1-z)^2+1\right)} \tag{154}$$

$$\tan^2\theta = -\frac{\left((1-z)^2-1\right)^2}{\left((1-z)^2+1\right)^2} \tag{155}$$

The derivative of a new variable z with respect to θ is

$$\frac{dz}{d\theta} = \frac{d\left(1-e^{i\theta}\right)}{d\theta} = -ie^{i\theta} = i(z-1) \tag{156}$$

The first derivative of the wave function Θ with respect to θ is

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{dz}\frac{dz}{d\theta} = i(z-1)\frac{d\Theta}{dz} \tag{157}$$

The second derivative of the wave function Θ with respect to θ is

$$\frac{d^2\Theta}{d\theta^2} = -(z-1)^2\frac{d^2\Theta}{dz^2} - (z-1)\frac{d\Theta}{dz} \tag{158}$$

We calculate the derivative of Θ in terms of a new function T

$$\frac{d\Theta}{dz} = \left(\rho z^{\rho-1}(1-z)^{\sigma} - \sigma z^{\rho}(1-z)^{\sigma-1}\right)T + z^{\rho}(1-z)^{\sigma}\frac{dT}{dz} \tag{159}$$

The second derivative is

$$\begin{aligned}\frac{d^2\Theta}{dz^2} &= \left[\left(\rho(\rho-1)z^{\rho-2}(1-z)^{\sigma} - 2\rho\sigma z^{\rho-1}(1-z)^{\sigma-1} + \sigma(\sigma-1)z^{\rho}(1-z)^{\sigma-2}\right)+\right]T \\ &\quad +2\left(\rho z^{\rho-1}(1-z)^{\sigma} - \sigma z^{\rho}(1-z)^{\sigma-1}\right)\frac{dT}{dz} + z^{\rho}(1-z)^{\sigma}\frac{d^2T}{dz^2}\end{aligned} \tag{160}$$

By substituting the results 154 to 158 in equation 149 we find a new angular equation

$$-(z-1)^2\frac{d^2\Theta}{dz^2} - (z-1)\frac{d\Theta}{dz} - \left(-\alpha\frac{\left((1-z)^2-1\right)^2}{\left((1-z)^2+1\right)^2} - i\beta\frac{\left((1-z)^2-1\right)}{\left((1-z)^2+1\right)} + \gamma\right)\Theta - E_{\theta}\Theta = 0 \tag{161}$$

By using the equations 151, 159 and 160 the equation 161 becomes

$$\begin{gathered}
-z^{\rho}(1-z)^{\sigma+2}\frac{d^2T}{dz^2}+\left[-2\left(\rho z^{\rho-1}(1-z)^{\sigma+2}-\sigma z^{\rho}(1-z)^{\sigma+1}\right)+z^{\rho}(1-z)^{\sigma}\right]\frac{dT}{dz}+\\
\left[-\rho\left(\rho-1\right)z^{\rho-2}(1-z)^{\sigma+2}+2\rho\sigma z^{\rho-1}(1-z)^{\sigma+1}-\right.\\
\sigma\left(\sigma-1\right)z^{\rho}(1-z)^{\sigma}-\rho z^{\rho-1}(1-z)^{\sigma+1}-\sigma z^{\rho}(1-z)^{\sigma}\\
\left.-\left(\alpha\left(\frac{-i\left((1-z)^2-1\right)}{\left((1-z)^2+1\right)}\right)^2\theta+\beta\frac{-i\left((1-z)^2-1\right)}{\left((1-z)^2+1\right)}+\gamma\right)z^{\rho}(1-z)^{\sigma}-E_{\theta}z^{\rho}(1-z)^{\sigma}\right]T=0
\end{gathered} \tag{162}$$

We divide by $z^{\rho-1}(1-z)^{\sigma+1}$we find

$$\begin{gathered}
-z(1-z)\frac{d^2T}{dz^2}+\left[-2\left(\rho(1-z)-\sigma z\right)+z(1-z)^{-1}\right]\frac{dT}{dz}+\\
\left[-\rho\left(\rho-1\right)z^{-1}(1-z)+2\rho\sigma-\sigma^2z(1-z)^{-1}-\rho-\right.\\
\left.\left(\alpha\left(\frac{-i\left((1-z)^2-1\right)}{\left((1-z)^2+1\right)}\right)^2\theta+\beta\frac{-i\left((1-z)^2-1\right)}{\left((1-z)^2+1\right)}+\gamma\right)E_{\theta}z(1-z)^{-1}-E_{\theta}z(1-z)^{-1}\right]
\end{gathered} \tag{163}$$

Where we put

$$\rho=\frac{1}{2}+\frac{1}{2}(1+4\alpha)^{1/2} \tag{164}$$

And

$$\sigma=\frac{1}{2}(-E_{\theta}+\alpha-i\beta-\gamma)^{1/2} \tag{165}$$

We get following hypergeometric equation

$$z(1-z)T^{"}+\left[-(2\rho+2\sigma+1)z\right]T^{'}-\left[2\rho\sigma+\rho+\alpha-\frac{i\beta}{2}\right]T=0 \tag{166}$$

The solution of this equation is hypergeometric function :[3][36]

$$T=F\left(2\rho,2\sigma,1+(1+4\alpha)^{1/2};z\right) \tag{167}$$

From the asymptotic behavior of the confluent series ($r\rightarrow\infty\Longrightarrow F=0$) which lead to $T\rightarrow0$ when $r\rightarrow\infty$ we find the general condition of quantization :

$$2\rho=-m\Longrightarrow2\rho+m=0,m=0,1,2,... \tag{168}$$

From the condition of hypergeometric equation we have

$$2\sigma=2\left(\frac{1}{2}(-E_{\theta}+\alpha-i\beta-\gamma)^{1/2}\right)=(-E_{\theta}+\alpha-i\beta-\gamma)^{1/2} \tag{169}$$

By using 168 we find

$$2\sigma = m + 2\rho + (-E_\theta + \alpha - i\beta - \gamma)^{1/2} \tag{170}$$

So we can write the hypergeometric function as

$$T = F(-m, m+1+(1+4\alpha)^{1/2} + (-E_\theta + \alpha - i\beta - \gamma)^{1/2}; 1+(1+4\alpha)^{1/2}; z) \tag{171}$$

The equation 170 give us the energy E_θ as

$$E_\theta = \alpha - i\beta - \gamma - 4\sigma^2 \tag{172}$$

From the form of the hypergeometric function we have

$$\sigma^2 = \frac{1}{4}\left\{-i\beta + \left\{\left[(1+4\alpha)^{1/2}+1+2m\right]^4 - 4\beta^2\right\}\left[(1+4\alpha)^{1/2}+1+2m\right]^{-2}\right\} \tag{173}$$

We substitute the last equation in 172 we find the expression of angular energy as

$$E_\theta = \alpha - \gamma - \frac{\left[(1+4\alpha)^{1/2}+1+2m\right]^4 - 4\beta^2}{4\left[(1+4\alpha)^{1/2}+1+2m\right]^2} \tag{174}$$

.2 Details of Non-Cenral Potentials in 3D Ordinary Space

Case3 $V_5(r,\theta) = \mu\left[-\frac{H}{r} + \frac{D_r}{r^2} + \frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cos^4\theta + \beta\cos^2\theta + \gamma)\sin^{-2}\theta\right]$

For this case the angular equation 2.15 becomes

$$\frac{d^2\Theta(\theta)}{d\theta^2} + \cot\theta\frac{d\Theta(\theta)}{d\theta} - \frac{m^2}{\sin^2\theta}\Theta(\theta) - \left(\alpha\cos^4\theta + \beta\cos^2\theta + \gamma\right)\sin^{-2}\theta\cos^{-2}\theta - E_\theta\Theta(\theta) = 0 \tag{175}$$

We make the following substitutions

$$\omega = \cos^2(\theta) \tag{176}$$

And

$$\Theta = \omega^\rho (1-\omega)^\sigma T \tag{177}$$

So we have to compute all parts of the angular equation by the new variable

$$\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - \omega \tag{178}$$

And

$$\cot\theta = \frac{\sqrt{\omega}}{\sqrt{1-\omega}} \tag{179}$$

The first derivative of Θ with respect to θ in terms of new variable ω is

$$\frac{d\Theta}{d\theta} = -\left[2\sqrt{\omega\,(1-\omega)}\right]\frac{d\Theta}{d\omega} \tag{180}$$

The second derivative of Θ with respect to θ in terms of new variable is

$$\frac{d^2\Theta}{d\theta^2} = [2-4\omega]\frac{d\Theta}{d\omega} + 4\omega\,(1-\omega)\frac{d^2\Theta}{d\omega^2} \tag{181}$$

The first derivative $\frac{d\Theta}{d\omega}$ in terms of new function T is

$$\frac{d\Theta}{d\omega} = \left(\rho\omega^{\rho-1}(1-\omega)^{\sigma} - \sigma\omega^{\rho}(1-\omega)^{\sigma-1}\right)T + \omega^{\rho}(1-\omega)^{\sigma}\frac{dT}{d\omega} \tag{182}$$

The second derivative $\frac{d^2\Theta}{d\omega^2}$ in terms of new function T is

$$\begin{aligned}\frac{d^2\Theta}{d\omega^2} &= \left[\left(\rho\,(\rho-1)\,\omega^{\rho-2}(1-\omega)^{\sigma} - 2\rho\sigma\omega^{\rho-1}(1-\omega)^{\sigma-1} + \sigma\,(\sigma-1)\,\omega^{\rho}(1-\omega)^{\sigma-1}\right)\right]T \\ &\quad +2\left(\rho\omega^{\rho-1}(1-\omega)^{\sigma} - \sigma\omega^{\rho}(1-\omega)^{\sigma-1}\right)\frac{dT}{d\omega} + \omega^{\rho}(1-\omega)^{\sigma}\frac{d^2T}{d\omega^2}\end{aligned} \tag{183}$$

By substituting the results 178 to 181 in equation 175 we find a new angular equation in terms of the variable ω

$$4\omega\,(1-\omega)\frac{d^2\Theta}{d\omega^2} + [2-6\omega]\frac{d\Theta}{d\omega} - \left[\frac{1}{1-\omega}\left(m^2+\alpha\omega+\beta+\frac{\gamma}{\omega}\right)+E_\theta\right]\Theta(\theta) = 0 \tag{184}$$

We use 182, 183 the last equation becomes

$$\begin{aligned}&4\omega\,(1-\omega)\left[\omega^{\rho}(1-\omega)^{\sigma}\frac{d^2T}{d\omega^2} + 2\left(\rho\omega^{\rho-1}(1-\omega)^{\sigma} - \sigma\omega^{\rho}(1-\omega)^{\sigma-1}\right)\frac{dT}{d\omega}+\right. \\ &\left.\left(\rho\,(\rho-1)\,\omega^{\rho-2}(1-\omega)^{\sigma} - 2\rho\sigma\omega^{\rho-1}(1-\omega)^{\sigma-1} + \sigma\,(\sigma-1)\,\omega^{\rho}(1-\omega)^{\sigma-1}\right)T\right] \\ &+(2-6\omega)\left[\omega^{\rho}(1-\omega)^{\sigma}\frac{dT}{d\omega} + \left(\rho\omega^{\rho-1}(1-\omega)^{\sigma} - \sigma\omega^{\rho}(1-\omega)^{\sigma-1}\right)T\right] - \\ &\left[\frac{1}{1-\omega}\left(m^2+\alpha\omega+\beta+\frac{\gamma}{\omega}\right)+E_\theta\right]\omega^{\rho}(1-\omega)^{\sigma}T = 0\end{aligned} \tag{185}$$

We divide by $4\omega^{\rho}(1-\omega)^{\sigma}$ we find

$$\omega\left(1-\omega\right)\frac{d^{2}T}{d\omega^{2}}+\left[\left(2\rho+\frac{1}{2}\right)-(2\rho+2\sigma+\frac{3}{2})\omega\right]\frac{dT}{d\omega}+$$
$$\Big[\rho\left(\rho-1\right)\omega^{-1}+\rho\left(\rho-1\right)+\sigma\left(\sigma-1\right)\omega+\frac{1}{2}\rho\omega^{-1}-\frac{1}{2}\sigma(1-\omega)^{-1}+\frac{3}{2}\sigma\omega(1-\omega)^{-1}-$$
$$\frac{1}{4\left(1-\omega\right)}\left(m^{2}+\alpha\omega+\beta+\frac{\gamma}{\omega}\right)-2\rho\sigma-\frac{E_{\theta}}{4}-\frac{3}{2}\rho T=0\Big] \tag{186}$$

Where we put

$$\rho=\frac{1}{4}+\frac{1}{4}\left(1+\gamma\right)^{1/2} \tag{187}$$

And

$$\sigma=\frac{1}{2}(m^{2}+\alpha+\beta+\gamma)^{1/2} \tag{188}$$

We get a hypergeometric equation

$$\omega(1-\omega)T^{"}+\left[\left(2\rho+\frac{1}{2}\right)-(2\rho+2\sigma+\frac{3}{2})\omega\right]T^{'}-\frac{1}{4}\left[E_{\theta}+8\rho\sigma+2\rho+2\alpha+2\gamma+m^{2}+\beta\right]T=0 \tag{189}$$

The solution is hypergeometric function :

$$T=N_{\theta}F(-l,l+1+\frac{1}{2}\left(1+\gamma\right)^{1/2}+(m^{2}+\alpha+\beta+\gamma)^{1/2};1+\frac{1}{2}\left(1+\gamma\right)^{1/2};\omega) \tag{190}$$

From the form of the hypergeometric equation

$$\frac{1}{4}\left[E_{\theta}+8\rho\sigma+2\rho+2\alpha+2\gamma+m^{2}+\beta\right]=\left(-2\rho\right)\left(2\rho+2\sigma\right) \tag{191}$$

This require that

$$E_{\theta}=\frac{1}{4}+\alpha+\left[2l+1+\frac{1}{2}\left(1+\gamma\right)^{1/2}+(m^{2}+\alpha+\beta+\gamma)^{1/2}\right]^{2} \tag{192}$$

We find the angular wave function when we substitute the function T in the equation $\Theta=\omega^{\rho}\left(1-\omega\right)^{\sigma}T$ as

$$\Theta\left(z\right)=N_{\theta}\omega^{\rho}\left(1-\omega\right)^{\sigma}F(-l,l+1+\frac{1}{2}\left(1+\gamma\right)^{1/2}+(m^{2}+\alpha+\beta+\gamma)^{1/2};1+\frac{1}{2}\left(1+\gamma\right)^{1/2};\omega) \tag{193}$$

We use $\omega = \cos^2\theta$,so

$$\Theta(z) = N_\theta (\cos\theta)^{2\rho} \theta \left(1-\cos^2\theta\right)^\sigma$$
$$F(-l, l+1+\frac{1}{2}(1+\gamma)^{1/2} + (m^2+\alpha+\beta+\gamma)^{1/2}; 1+\frac{1}{2}(1+\gamma)^{1/2}; \cos^2\theta) \tag{194}$$

Where $\rho = \frac{1}{4}+\frac{1}{4}(1+\gamma)^{1/2}$,and $\sigma = \frac{1}{2}(m^2+\alpha+\beta+\gamma)^{1/2}$
And the angular energy is

$$E_\theta = \frac{1}{4}+\alpha+\left[2l+1+\frac{1}{2}(1+\gamma)^{1/2}+(m^2+\alpha+\beta+\gamma)^{1/2}\right]^2 \tag{195}$$

$l = 0, 1, 2, ..., m = 0, \pm1, \pm2, ...,$

Case5 $V_9(r,\theta) = \mu\left[\frac{H}{r}+\frac{D_r}{r^2}+\frac{1}{r^2}\left(\frac{\hbar^2}{2\mu^2}\right)(\alpha\cot^2\theta+\beta\cot\theta+\gamma)\right]$

For this case the angular equation 2.15 becomes

$$\frac{d^2\Theta(\theta)}{d\theta^2}+\cot\theta\frac{d\Theta(\theta)}{d\theta}-\frac{m^2}{\sin^2\theta}\Theta(\theta)-\left(\alpha\cot^2\theta+\beta\cot\theta+\gamma\right)\Theta(\theta)-E_\theta\Theta(\theta)=0 \tag{196}$$

To solve this equation we have to make the following substitutions:

$$z = e^{2i\theta} \Longrightarrow \theta = -\frac{i}{2}\ln(z) \tag{197}$$

And

$$\Theta = z^\rho(1-z)^\sigma T \tag{198}$$

From 197 we have

$$\sin^2\theta = \frac{-(1-z)^2}{4z} \tag{199}$$

And

$$\cot\theta = \frac{i(1+z)}{1-z} \Longrightarrow \cot^2\theta = \frac{-(1+z)^2}{(1-z)^2} \tag{200}$$

$$\frac{dz}{d\theta} = \frac{d\left(e^{2i\theta}\right)}{d\theta} = 2ie^{2i\theta} = 2iz \tag{201}$$

The first derivative of Θ with respect to θ in terms of new variable z is

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{dz}\frac{dz}{d\theta} = 2iz\frac{d\Theta}{dz} \tag{202}$$

The second derivative of Θ with respect to θ in terms of new variable z is

$$\frac{d^2\Theta}{d\theta^2} = -4z^2\frac{d^2\Theta}{dz^2} - 4z\frac{d\Theta}{dz} \tag{203}$$

We calculate the derivative of Θ with respect to z in terms of the new function T ,the first derivative is

$$\frac{d\Theta}{dz} = \left(\rho z^{\rho-1}(1-z)^{\sigma} - \sigma z^{\rho}(1-z)^{\sigma-1}\right) T + z^{\rho}(1-z)^{\sigma}\frac{dT}{dz} \tag{204}$$

The second derivative is

$$\begin{aligned}\frac{d^2\Theta}{dz^2} &= \left[\left(\rho\left(\rho-1\right) z^{\rho-2}(1-z)^{\sigma} - 2\rho\sigma z^{\rho-1}(1-z)^{\sigma-1} + \sigma\left(\sigma-1\right) z^{\rho}(1-z)^{\sigma-2}\right) + \right] T \\ &+2\left(\rho z^{\rho-1}(1-z)^{\sigma} - \sigma z^{\rho}(1-z)^{\sigma-1}\right)\frac{dT}{dz} + z^{\rho}(1-z)^{\sigma}\frac{d^2T}{dz^2}\end{aligned} \tag{205}$$

By substituting the results 199 to 203 in equation 196 we find a new form of angular equation

$$\begin{gathered}-4z^2\frac{d^2\Theta}{dz^2} - \left(4z + 2z\frac{(1+z)}{1-z}\right)\frac{d\Theta}{dz} + \\ \left(\frac{m^2 4e^{i2\theta}}{\left(1-e^{i2\theta}\right)^2} + \alpha\frac{(1+z)^2}{(1-z)^2} - i\beta\frac{(1+z)}{1-z} + \gamma - E_{\theta}\right)\Theta(\theta) = 0\end{gathered} \tag{206}$$

By using the equations 204 and 205 the equation 206 becomes

$$\begin{gathered}-4z^2\left[z^{\rho}(1-z)^{\sigma}\frac{d^2T}{dz^2} + 2\left(\rho z^{\rho-1}(1-z)^{\sigma} - \sigma z^{\rho}(1-z)^{\sigma-1}\right)\frac{dT}{dz} + \right. \\ \left.\left(\left(\rho\left(\rho-1\right) z^{\rho-2}(1-z)^{\sigma} - 2\rho\sigma z^{\rho-1}(1-z)^{\sigma-1} + \sigma\left(\sigma-1\right) z^{\rho}(1-z)^{\sigma-2}\right)\right) T\right] - \\ \left(4z + 2z\frac{(1+z)}{1-z}\right)\left[\left(\rho z^{\rho-1}(1-z)^{\sigma} - \sigma z^{\rho}(1-z)^{\sigma-1}\right) T + z^{\rho}(1-z)^{\sigma}\frac{dT}{dz}\right] + \\ \left[\frac{m^2 4e^{i2\theta}}{\left(1-e^{i2\theta}\right)^2} + \alpha\frac{(1+z)^2}{(1-z)^2} - i\beta\frac{(1+z)}{1-z} + \gamma - E_{\theta}\right] z^{\rho}(1-z)^{\sigma} T = 0\end{gathered} \tag{207}$$

We devise by $[-4z^{\rho+1}(1-z)^{\sigma-1}]$ we fin

$$
\begin{aligned}
&z(1-z)\frac{d^2T}{dz^2}+\left[\left(2\rho+\frac{1}{2}\right)-\left(2\rho z+2\sigma+\frac{3}{2}\right)z\right]\frac{dT}{dz}+\\
&\left[\left(\rho\left(\rho-1\right)z^{-1}(1-z)-2\rho\sigma+\sigma\left(\sigma-1\right)z(1-z)^{-1}\right)+\right.\\
&\left.\left(\rho z^{-1}(1-z)-\sigma\right)+\frac{1}{2}\frac{(1+z)}{1-z}\left(\rho z^{-1}(1-z)-\sigma\right)\right]T\\
&+\frac{1}{4}\left[\frac{m^2 4}{(1-z)}-\alpha\frac{z^{-1}\left(1+z\right)^2}{(1-z)}+i\beta\left(1+z\right)z^{-1}+\gamma-E_\theta\right]T=0
\end{aligned}
\tag{208}
$$

Where we put

$$
\rho=\frac{1}{4}+\frac{1}{2}(\frac{1}{4}-\gamma-E_\theta+i\beta+\alpha)^{1/2} \tag{209}
$$

And

$$
\sigma=(m^2+\alpha)^{1/2} \tag{210}
$$

We get a hypergeometric equation

$$
z(1-z)\frac{d^2T}{dz^2}+\left[\left(2\rho+\frac{1}{2}\right)-\left(2\rho z+2\sigma+\frac{3}{2}\right)z\right]\frac{dT}{dz}-\left[2\rho\sigma+\rho+\alpha-\frac{i\beta}{2}\right]T=0 \tag{211}
$$

The solution is hypergeometric function :

$$
T=F(-l,l+1+(\frac{1}{4}-\gamma-E_\theta+i\beta+\alpha)^{1/2}+2(m^2+\alpha)^{1/2};1+(\frac{1}{4}-\gamma-E_\theta+i\beta+\alpha)^{1/2};z) \tag{212}
$$

From the form of the hypergeometric equation

$$
4\rho\sigma=-\frac{1}{2}\left[2E_\theta+\rho+\sigma+4\rho\sigma+\gamma-\alpha\right]\Longrightarrow 8\rho\sigma=-2E_\theta-\rho-\sigma-4\rho\sigma-\gamma+\alpha \tag{213}
$$

This require that

$$
E_\theta=\frac{1}{4}-\gamma+\alpha-\frac{\left(2l+1+2\sqrt{m^2+\alpha}\right)^4-4\beta^2}{4\left(2l+1+2\sqrt{m^2+\alpha}\right)^2} \tag{214}
$$

$l=0,1,2,.....$and $m=0,\pm1,\pm2,.....$

We find the angular wave function when we substitute the function T in equation 198 as

$$\Theta\left(y\right)=z^{\rho}(1-z)^{\sigma}$$
$$N_{\theta}F(-l,l+1+(\frac{1}{4}-\gamma-E_{\theta}+i\beta+\alpha)^{1/2}+2(m^{2}+\alpha)^{1/2};1+(\frac{1}{4}-\gamma-E_{\theta}+i\beta+\alpha)^{1/2};z) \tag{215}$$

We use $z=e^{2i\theta}$,so

$$\Theta\left(z\right)=N_{\theta}e^{i2\rho\theta\rho}(1-e^{2i\theta})^{\sigma}$$
$$F(-l,l+1+(\frac{1}{4}-\gamma-E_{\theta}+i\beta+\alpha)^{1/2}+2(m^{2}+\alpha)^{1/2};1+(\frac{1}{4}-\gamma-E_{\theta}+i\beta+\alpha)^{1/2};e^{2i\theta}) \tag{216}$$

Where $\rho=\frac{1}{4}+\frac{1}{4}\left(1+\gamma\right)^{1/2}$,$\rho=\frac{1}{4}+\frac{1}{2}(\frac{1}{4}-\gamma-E_{\theta}+i\beta+\alpha)^{1/2}$ and $\sigma=(m^{2}+\alpha)^{1/2}$

Afterword

The works concern to this subject which we can study it in future are

- The effect of gravity on the non-central potentials
- The non-central potentials in the formalism of non commutative geometry

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Abstract

In this thesis we conducted a quantum study of non-central potentials that can be studied analytically, where we addressed the solvable potentials in the 2D ordinary space where we addressed both the relativistic and non-relativistic states and as a results of the study we concluded the energy spectrum and the wave function of a charged particle circulating in these potentials ,in this case we studied in detail the Kratzer potential plus dipole potential, then the pseudoharmonic oscillator potential plus dipole potential. In the second stage, with the same previous study, we dealt with other potentials that can be solved in the 3D ordinary space, where in this case we studied in detail the ring-shaped potential plus Kratzer potential, then the ring-shaped potential plus the pseudoharmonic oscillator potential. In the final stages, we treated the same potentials with the same dimensions, but in the deformed space (de Sitter and anti-de Sitter) where we noted the influence of the deformation coefficient on the energy spectrum and its effect on the wave function, then we deduced the critical values of the deformation coefficient for the existence of bound states

Keywords: Schrödinger equation, Non-Central Potentials, de Sitter and anti-de Sitter Space

ملخص

في هذه الاطروحة قمنا باجراء دراسة كمية للكمونات اللامركزية التي يمكن دراستها تحليليا حيث عالجنا الكمونات القابلة للحل في الفضاء الاعتيادي ثنائي البعد اين تناولنا كلتا الحالتين النسبية و غير النسبية و كنتيجة للدراسة استنتجنا طيف الطاقة ودالة الموجة للنظام المتكون من جسم مشحون يدور في هذه الكمونات،وفي هذه الحالة درسنا بالتفصيل كمون كراتزر زائد كمون ثنائي القطب ثم كمون الهزاز التوافقي المستعارزائد كمون ثنائي القطب،في المرحلة الثانية تناولنا بنفس الدراسة كمونات اخرى قابلة للحل في الفضاء الاعتيادي ثلاثي البعد حيث في هذه الحالة درسنا بالتفصيل كمون الحلقة زائد كمون كراتزر ثم كمون الحلقة زائد كمون الهزاز التوافقي المستعارفي المراحل الاخيرة عالجنا نفس الكمونات وبنفس الابعاد لكن في الفضاء المشوه (فضاء دي سيتر و ضد دي سيتر) حيث لاحضنا تأثير معامل التشوه على طيف الطاقة للنظام السابق وتأثيره على دالة الموجة واستنتجنا القيم الحرجة لمعامل التشوه التي تجعل وجود حالات مرتبطة للنظام

كلمات مفتاحية: معادلة شرودينغر،الكمونات اللامركزية ،فضاء دي سيتر وضد دي سيتر

Abstrait

Dans cette thèse, nous avons mené une étude quantique des potentiels non-centraux qui peuvent être étudiés analytiquement, où nous avons abordé les potentiels résolubles dans l'espace 2D ordinaire dans le cas relativiste et non relativistes, à la suite de l'étude, nous avons conclu le spectre énergétique et la fonction d'onde d'un corps chargé circulant dans ces potentiels, dans ce cas, nous avons étudié en détail le potentiel de Kratzer plus le potentiel dipolaire, puis le potentiel d’oscillateur pseudoharmonique plus le potentiel dipolaire. Dans la deuxième étape, avec la même étude précédente, nous avons traité d'autres potentiels qui peuvent être résolus dans l'espace 3D ordinaire, où dans ce cas nous avons étudié en détail le potentiel d'anneau plus le potentiel de Kratzer, puis le potentiel d'anneau plus le potentiel d’oscillateur pseudoharmonique. Dans les étapes finales, nous avons traité les mêmes potentielles avec les mêmes dimensions, mais dans l'espace déformé (de Sitter et anti-de Sitter) où nous avons remarqué l'influence du coefficient de déformation sur le spectre d'énergie et son effet sur la fonction d’onde, puis nous avons déduit les valeurs critiques du coefficient de déformation pour l'existence des états liés

Mots clés : l’équation de Schrödinger, les potentiels non-centraux, l’espace de Sitter et anti-de Sitter